Algebra and Simultaneous Equations Story

The name algebra is derived from the title of the ground breaking book, *Kitab al-mukhtasar fi hisab* ***aljabr*** *wa'l-muqabala,* written by a Muslim scholar Muhammad ibn Musa al-Khwarizmi, in 825 AD. A systematic study of methods for solving quadratic equations constituted a central concern of Muslim mathematicians, and hence their important contributions to the progress of algebraic thinking. A no less central contribution is related to the Muslim reception and transmission of ideas related to the Hindu system of numeration, to which they also added fundamental components lacking so far, such as decimal fractions.

Al-Khwarizmi's work embodies much of what is central to Muslim contributions in this field. He declared his book to be intended as one of practical value, yet this definition hardly applies to what one finds there. In the first part of his book Al-Khwarizmi presented the procedures for solving six types of equations: squares equal roots, squares equal numbers, roots equal numbers, squares and roots equal numbers, squares and numbers equal roots, and roots and numbers equal a square. (ln modern notation: $ax^2 = bx$, $ax^2 = c$, $bx = c$, $ax^2 + bx = c$, $ax^2 + c = bx$, and $bx + c = ax^2$, respectively.) Neither zero nor the negative numbers appear here as legitimate coefficients or solutions to equations. Moreover, we find nothing like symbolic representation or abstract symbol manipulation and, in fact in the problems, even the quantities are written in words rather than in symbols. All procedures are described verbally. This is nicely illustrated by the following, typical example:

> *What must be the square which, when increased by ten of its own roots, amounts to thirty- nine? The solution is this: You halve the number of roots, which in the present instance yields five. This you multiply by itself; the product is twenty-five. Add this to thirty-nine; the sum is sixty-four. Now take the root of this which is eight, and subtract form it half the number of the roots which is five; the remainder is three. This is the root of the square which you sought for.*

In the second part, al-Khwarizmi uses propositions taken from book II of Euclid's *Elements* in order to provide geometrical justification for his procedures. As remarked above, in their original context those were purely geometrical texts. Here they are directly connected, for the first time, to the question of solving quadratic equations. This is a hallmark of the Muslims' approach to solving equations: systematize all cases and provide a geometric justification, based on Greek sources. In the XI century, the Persian mathematician, astronomer, and poet Omar Khayyam showed how to express roots of cubic equations by line segments obtained by intersecting conic sections. Omar Khayyam applied Greek knowledge on conic sections to questions involving cubic variables.

The use of Greek-style, geometrical arguments in this context also led to a gradual loosening of certain basic, traditional constraints. Thus, Muslim mathematicians allowed, and indeed encouraged at variance with the Greek tradition, the unrestricted combination of commensurable and incommensurable magnitudes within the same framework, as well as the simultaneous manipulations of magnitudes of different dimensions as part of the solution of an individual problem. Thus, in the work of Abu Kamil the solution of a quadratic equation is a "number", rather than a "line segment" or an "area". Combined with the use of the decimal system, this approach was fundamental in developing a more abstract and general conception of number, which eventually became essential for the creation of a full-fledged abstract idea of an equation.

The beginning of algebra goes back to the time of ancient Egypt and Babylon, where people learned to solve linear (ax = b) and quadratic (ax2 + bx = c) equations, as well as indeterminate equations such as x2 + y2 = z2, whereby several unknowns are involved. The ancient Babylonians solved arbitrary quadratic equations by essentially the same procedures taught today. They also could solve some indeterminate equations.

The Alexandrian mathematicians Hero of Alexandria and Diophantus continued the traditions of Egypt and Babylon, but Diophantus's book Arithmetica is on a much higher level and gives many surprising solutions to difficult indeterminate equations.

Ancient civilizations wrote out algebraic expressions using only occasional abbreviations, but by medieval times Muslim mathematicians were able to talk about arbitrarily high powers of the unknown x, and work out the basic algebra of polynomials (without yet using modern symbolism). This included the ability to multiply, divide, and find square roots of polynomials as well as a knowledge of the binomial theorem.

Indian mathematicians such as Brahmagupta and Bhaskara in 6-12th developed non-symbolic, yet very precise, procedures for solving equations of degree one and two, and equations on more than one variable. However, the main contribution of Hindu mathematics to algebra concerns the elaboration of the decimal, positional numeral system, which closely accompanied the development of symbolic algebra in renaissance Europe. By the ninth century the Hindus certainly had a full-fledged decimal, positional system, yet many of its central ideas had been transmitted well before that to China and the Islam. Hindu arithmetic, moreover, developed consistent and correct sets of rules for operating with positive and negative numbers, and zero was treated as a number like any other, even in problematic contexts such as division. It would still take several hundreds of years before European mathematics would be in a position to fully integrate ideas of this kind into the developing discipline of algebra.

Chinese mathematicians during the period parallel to the European middle ages developed their own methods for solving quadratic equations by "radicals" (i.e.: displaying the solutions as expression involving the coefficients, the four basic algebraic operations, and roots of them) and for classifying such solutions. They also attempted to solve higher degree equations in this same direction, yet unsuccessfully. Thus, they were led to approximation methods of high accuracy, such as developed by Yang Hui in the twelfth century AD. The advantages in calculations afforded by their expertise with the abacus may help explain why Chinese mathematicians followed more intensively this approach rather than make additional progress with radical methods.

Early in the 16th century, the Italian mathematicians Scipione del Ferro, Niccolò Tartaglia, and Gerolamo Cardano solved the general cubic equation in terms of the constants appearing in the equation. Cardano's pupil, Ludovico Ferrari, soon found an exact solution to equations of the fourth degree (see quartic equation), and as a result, mathematicians for the next several centuries tried to find a formula for the roots of equations of degree five, or higher. Early in the

19th century, however, the Norwegian mathematician Niels Abel and the French mathematician Evariste Galoisproved that no such formula exists.

An important development in algebra in the 16th century was the introduction of symbols for the unknown and for algebraic powers and operations. As a result of this development, Book III of La géometrie (1637), written by the French philosopher and mathematician René Descartes, looks much like a modern algebra text. Descartes's most significant contribution to mathematics, however, was his discovery of analytic geometry, which reduces the solution of geometric problems to the solution of algebraic ones. His geometry text also contained the essentials of a course on the theory of equations, including his so-called rule of signs for counting the number of what Descartes called the "true" (positive) and "false" (negative) roots of an equation. Work continued through the 18th century on the theory of equations, but not until 1799 was the proof published, by the German mathematician Carl Friedrich Gauss, showing that every polynomial equation has at least one root in the complex plane.

By the time of Gauss, algebra had entered its modern phase. Attention shifted from solving polynomial equations to studying the structure of abstract mathematical systems whose axioms were based on the behaviour of mathematical objects, such as complex numbers, that mathematicians encountered when studying polynomial equations. Two examples of such systems are algebraic groups and quaternions, which share some of the properties of number systems but also depart from them in important ways. Groups began as systems of permutations and combinations of roots of polynomials, but they became one of the chief unifying concepts of 19th-century mathematics. Important contributions to their study were made by the French mathematicians Galois and Augustin Cauchy, the British mathematician Arthur Cayley, and the Norwegian mathematicians Niels Abel and Sophus Lie. Quaternions were discovered by British mathematician and astronomer William Rowan Hamilton, who extended the arithmetic of complex numbers to quaternions while complex numbers are of the form a + bi, quaternions are of the form a + bi + cj + dk.

Immediately after Hamilton's discovery, the German mathematician Hermann Grassmann began investigating vectors. Despite its abstract character, American physicist J. W. Gibbs recognized in vector algebra a system of great utility for physicists, just as Hamilton had recognized the usefulness of quaternions. The widespread influence of this abstract approach led George Boole to write The Laws of Thought (1854), an algebraic treatment of basic logic. Since that time, modern algebra—also called abstract algebra—has continued to develop. Important new results have been discovered, and the subject has found applications in all branches of mathematics and in many of the sciences as well.

The classical discipline of algebra starts its actual development after the consolidation of the idea of an equation in Viete’s work. At the same time, during this same period of time, new mathematical objects gradually arose (groups, rings, fields, etc.), that eventually came to replace the study of polynomials as the main subject matter of algebra and became the new focus of interest of the discipline.

The creation of what came to be known as analytic geometry is usually attributed to two famous French thinkers: Pierre de Fermat (1601-1665) and Rene Descartes (1596-1650). They used the algebraic techniques developed by Cardano and Viete and applied them to tackle classical geometrical problems that had remained unsolved since the time of the Greeks. The new kind of organic connection between algebra and geometry thus established meant a major breakthrough without which the subsequent development of mathematics in general, and in particular of geometry and the calculus, would be unthinkable. It also had significant impact on algebraic thinking.

In his famous book *La Geometrie* (1637) Descartes established equivalences between algebraic operations and geometrical constructions. In order to do so, he introduced a "unit length", serving as reference for all other lengths and all operations among them. Descartes found the square root of any given number, as represented by a line segment. However, the key step in the construction has been the introduction of the "unit length" Functional Groups (FG). This seemingly trivial move, or anything similar to it, had never been part of Greek geometry and its legacy and, of course, it had enormous repercussions on what could now be done by applying algebraic reasoning to geometry.

Descartes' work was a starting point for the definite transformation of polynomials into an autonomous object of intrinsic mathematical interest. Algebra became identified, to a large extent, with the theory of polynomials. A clear notion of a polynomial equation, together with existing techniques for solving some of them, allowed for a coherent and systematic reformulation of many questions that mathematicians in the past had dealt with in a more haphazard fashion. High in the agenda remained the open question of finding algebraic solutions of equations of degree higher than four. Closely related to this was the question of the kinds of numbers that should count as legitimate roots of equations. The attempts to deal with these two important problems helped realize the centrality of another pressing question that needed to be elucidated, namely, the questions of the number of solutions that a given polynomial equation has.

The answer to this question is afforded by the so-called Fundamental Theorem of Algebra (FTA), which asserts that every polynomial in real coefficients can be expressed as the product of linear and quadratic (real) factors, or, alternatively, that every polynomial equation of degree n in complex coefficients has n complex roots.

The first complete proof of the FTA is usually attributed to Carl Friedrich Gauss (1777- 1855) in his doctoral dissertation of 1799. Subsequently, Gauss himself provided three additional proofs. Later on, additional proofs were given by others, such as the Swiss bookkeeper Jean-Robert Argand (1768-1822) in 1814, and the German mathematician father and son, Hellmuth Kneser (1898-1973) in 1940 and Martin Kneser in 1981.

A major breakthrough in the way to elucidating the question of algebraically solving higher-degree equations was achieved by Lagrange in 1770. Rather than trying to directly find a possible solution for an equation of degree five, Lagrange attempted to clarify first *why* all attempts to do so had failed so far. He investigated the known solutions of cubic (i.e.: third-degree) and cuartic (i.e.: fourth-degree polynomial) equations and in particular, how certain algebraic expressions connected with those solutions remain invariant when the coefficients of the equations are permuted with one another. Lagrange was certain that a deeper analysis of this invariance would provide the key insight to understanding the essence of existing methods of solution by radicals, in the hope of being then able to extend them successfully to higher degrees.

Using the ideas developed by Lagrange, the Italian Paolo Ruffini (1765-1822) was the first mathematician ever to assert the impossibility of an algebraic solution for the *general* polynomial equation of degree greater than four. He adumbrated in his work the notion of a group of permutations (see below), and worked out some of its basic properties. Ruffini 's proofs, however, contained several, significant gaps.

The Norwegian mathematical star of the early nineteenth century, Niels Henrik Abel (1802-1829), provided in 1824 the first clear and accepted proof of the impossibility of solving by radicals, equations of degree five or above. This did not bring the question to an end, but rather opened an entirely new field of research, since, as Gauss 's example showed, *some* equations were indeed solvable.

Rather than establishing for specific equations if they can or cannot be solved by radicals, as Abel had suggested, Evariste Galois (1811-1832) pursued the somewhat more general problem of defining necessary and sufficient conditions for the solvability of any given equation.

A series of unusual and unfortunate events involving the most important French contemporary mathematicians prevented Galois' ideas from being published for a long time. It was not until 1846 that Joseph Liouville (1809-1882) edited and published for the first time, in his prestigious *Journal de Mathematiques Pures et Appliquees*, the important memoire where Galois had presented his main ideas and that the Paris Academy had turned down in 1831. Liouville also lectured in Paris on the topic, to a reduced audience. Leopold Kronecker (1823-1891) working in Berlin, applied some of these ideas to number theory in 1853, and Richard Dedekind (1831-1916) lectured on Galois theory in 1856 at G6titngen. At this point, however, the impact of the theory was still minimal.

A major turning-point came with the works of the leading Paris mathematician Camille Jordan (1838-1922) who published a series of papers and an influential book in 1870. Jordan elaborated a theory of groups of permutations, independently of any reference to equations, and the use of this theory to the question of algebraic solvability appeared in his book just as a particular application of the theory. A lengthy process eventually led from here to the conception of Galois theory as the study of the interconnections between extensions of fields and the related Galois groups of equations, a conception that would prove fundamental for developing a completely new approach to algebra in the 1920s. Major contributions to the development of this point of view for Galois theory came variously from later works by Dedekind, Enrico Betti (1823-1892), Henrich Weber (1842-1913), and Emil Artin (1898-1962), among others.

Felix Klein (1849-1925) was still a very young professor when in his inaugural lecture at the University of Erlangen (1872) he suggested how group theoretical ideas might be

fruitfully put to use in the context of geometry. Since the beginning of the 19th century the study of projective geometry had attained renewed impetus, and later on, non-Euclidean geometries were introduced and increasingly investigated. This proliferation of geometries raised pressing questions concerning both the interrelations among them and their relations with the empirical world.

Klein suggested that the many kinds of existing geometries could be classified and ordered within a conceptual hierarchy: thus, for instance, projective geometry seems to be more fundamental, because projective properties are relevant also, e.g., in Euclidean geometry. The main concepts of the latter, however, such as length or angle, have no significance in the former. But then, this hierarchy may be expressed in terms of transformations that leave invariant such properties as are distinctly relevant to each of the geometries in question. These transformations, it turns out, are best understood when seen as forming a group. An example related with Euclidean geometry clearly illustrates the basic idea behind this.

In the 1880's and 1890's, Klein's friend, the Norwegian Sophus Lie (1842-1899) undertook, together with some of his students at Leipzig, the enormous task of classifying all possible groups of continuous groups of geometric transformations, a task that would eventually evolve into the modern theory of Lie groups and Lie algebras. At roughly the same time, Jules Henri Poincaré (1854-1912) studied in France the groups of motions of rigid bodies, a work that contributed more than the others mentioned here to spreading the notion of group as a main tool in modern geometry.

The notion of group also started to appear prominently in number theory in the nineteenth century, especially in the work of Gauss on modular arithmetic. In this context he proved results that were later generally reformulated in the abstract theory of groups. Thus, for instance (in modern terms), that in a cyclic group there always exists a subgroup of every order dividing the order of the group. Gauss also studied the group-theoretical properties of transformations of quadratic forms, forms that play a major role in his number-theoretical investigations.

Arthur Cayley (1821-1895), one of the most prominent British mathematicians of his time, was the first to explicitly realize, in 1854, that a group could be defined abstractly, i.e.: without any reference to the nature of its elements and only by specifying the properties of the operation defined on them.

In 1854, even the idea of group of permutations was rather new and thus Cayley's work had little impact. It would take until 1882, and several additional articles by Cayley himself, as well as by Eugene Netto (1846-1919) and Georg Frobenius (1849-1917), before Walther van Dyck (1856-1934) would publish in 1882 the full-fledged and most general definition of an abstract group. Books like Heinrich Weber's *Lehrbuch de Algebra* (1895) and *Theory of Groups* (1897) by William Burnside (1852-1927) were instrumental in bringing the theory to a truly broad audience of mathematicians.

In spite of the many novel ideas that arose in connection with algebra in the nineteenth century, solving equations and studying properties of polynomial forms continued to be the main focus of interest of the discipline. An important offshoot of the study of polynomials was the development of the theory of algebraic invariants, to which much effort was dedicated by leading algebraists since the 1840s, especially in Germany (but which, for lack of space will not be considered here). The study of systems of equations led to developing the notion of a determinant and, later on, to the theory of matrices.

Given a system of *n* linear equations in *n* unknowns, a determinant is the result of a certain combined multiplication and addition of the coefficients involved, that allows calculating directly the values of the unknowns. Thus, for instance, given the system:

$a_1x + b_1y = c_1$
$a_2x + b_2y = c_2$

The determinant of the system is the number $\Delta = a_1.\, b_2 - a_2b_1$, and the values of the unknowns are given by

$x = (c_1.b_2 - c_2.b_1)/\Delta, y = (a_1.c_2 - a_2.c_1)/\Delta$

Cauchy published in 1815 the first truly systematic and comprehensive study of determinants (including the very name). He introduced the notation (a_1, n) for the system of coefficients of the system and showed how to calculate the value of the determinant by expanding any row or column with the adjoint of every element.

Closely related with determinants is the idea of a matrix, namely, any arrangement of numbers in lines and columns. That such an arrangement can be taken as an *autonomous mathematical object,* on which one can define a special arithmetic and operate as with ordinary numbers, was first conceived by Cayley and his good friend James Joseph Sylvester (1814-1897), in the 1850s. Determinants were a main, direct source for this idea, but so were ideas contained in previous work on number theory by Gauss and by Ferdinand Gotthold Eisenstein (1823-1852).

David Hilbert (1862-1943) was the most influential German mathematician of the turn of the century, and a leading algebraist as that. His early work on algebraic invariants reshaped this sub-discipline, through a legitimization of non-constructive proofs for the existence of certain algebraic objects (a finite basis of a system of invariants, in this case). His work on the theory of algebraic number fields in the 1890s was decisive in establishing the conceptual approach promoted by Dedekind, in opposition to the more algorithmically oriented one of Kronecker, as the dominant one in the discipline for the next decades. His work on the foundations of geometry, starting on 1899, introduced a totally new approach to the use of axiomatically defined concepts in mathematics at large. The undisputed leader of the vibrant world-class center of exact sciences in G6ttingen, his influence was enormously felt through the 68 doctoral dissertations he directed, as well as through the tens of distinguished mathematicians that started their careers as students under his spell. The structural view of algebra was to a large extent the product of some of Hilbert’s innovations, yet Hilbert himself basically remained a representative of the classical discipline of algebra. It is likely that the kind of algebra that developed under the influence of van der Waerden’s book was of no direct appeal to Hilbert.

In 1910 Ernst Steinitz (1871-1928) published one of the most influential milestones leading to the structural image of algebra in a research piece on the abstract theory of fields. The greatest influence behind the consolidation of the structural image of algebra is no doubt Emmy Noether, who became the most prominent figure in Gottingen in the 1920s.

After the late 1930s it was clear that algebra, and in particular the structural approach within it, had become a most dynamic domain of research, and its methods, results and concepts were being actively pursued by mathematicians in Germany, France, the USA, Japan and others. It was also successfully applied to redefine several classical mathematical disciplines. Two important early examples of this are the thorough reformulation of algebraic geometry in the hands of Van der Waerden, Weil, and Oscar Zariski (1899- 1986), using the concepts and the approach developed in ring theory by Emmy Noether and their successors, and the work of Marshall Stone (1903-1989), who in the late 1930s defined Boolean algebras, bringing under a purely algebraic framework ideas stemming from logic, topology and algebra itself.

Over the following decades several additional textbooks in algebra appeared following the paradigm established by van der Waerden. Prominent among these is *A Survey of Modern Algebra* first published in 1941 by Saunders Mac Lane (1909 -) and Garret Birkhoff (1921-1996), a book that became fundamental to the next several generations of the thriving algebraic research community in the USA. Algebra was increasingly taught and investigated now from a structural perspective all around the world.

Neverthles, it is important to stress that not all algebraists felt, at least at the beginning, that the new direction implied by *Moderne Algebra* was the correct one to follow. A much more classically-oriented research with deeply significant results in group theory, theory of group representations, Lie groups, etc. was still being carried out until well into the 1930s and much later. Worth of special attention in this respect are, among many others, Georg Frobenius, and Issai Schur (1875-1941), who were the most outstanding representatives of the Berlin mathematical school at the beginning of the century, and together with them, one of Hilbert's most prominent students, Hermann Weyl (1895-1955).

Embedded Story
Simultaneous Equations Story

A set of two or more equations, each containing two or more variables whose values can simultaneously satisfy both or all of the equations in the set., the number of variables being equal to or less than the number of equations in the set.

The linear equations are said to be simultaneous if they are considered at the same time. for example, x + y = 5 and x - y = 1 are simultaneous when considered together.

For a system of equations to have a unique solution, the number of equations must equal the number of unknowns. Even then a solution is not guaranteed. If a solution exists, the system is consistent; if not, it is inconsistent. A system of linear equations can be represented by a matrix whose elements are the coefficients of the equations. Though simple systems of two equations in two unknowns can be solved by substitution, larger systems are best handled with matrix techniques.

Example 1. Ahmad has more money than Ali. If Ahmad gave Ali INR 20, they would have the same amount. While if Ali gave Ahmad INR 22, Ahmad would then have twice as much as Ali. How much does each one actually have?

Solution Pointers

Let Ahmad have x, and let Ali have y.

First Action: x-20=y=20, Therefore x-y=40

Second Action: x+22=2*(y-22), Therefore, x-2y=-66

MATLAB:

```
>> a1=[1; 1]
a1 =
     1
     1

>> a2=[-1; -2]
a2 =
    -1
    -2

>> A=[a1 a2]
A =
     1    -1
     1    -2

>> b=[40; -66]
b =
    40
   -66
>> s=A\b
s =
   146
   106
```

Therefore, Ahmad started with INR 146 and Ali with INR 106

Example 2. 1000 tickets were sold. Adult tickets cost INR 8.50, children's cost INR 4.50, and a total of INR 7300 was collected. How many tickets of each kind were sold?

Solution Pointers.

Let *x* be the <u>*number*</u> *of adult tickets*. Let *y* be the <u>*number*</u> *of children's tickets*.

Therefore, x + y=1000 and 8.5x + 4.5y = 7300

MATLAB:

```
>> A=[1 1; 8.5 4.5]
A =
    1.0000    1.0000
    8.5000    4.5000

>> b=[1000; 7300]
b =
        1000
        7300
>> s=A\b
```

```
s =
  700
  300
```

Therefore, number of adult tickets sold is 700 and children 300

Example 3. Mrs. Puri invested INR 30,000 in two stocks. The first gave a dividend of 5% and the second stock 8%. The total dividend on the investment was INR 2,100. How much did she invest in each stock?

Solution Pointers

Total invest: x + y = 30000

Total dividend: .05x + .08y = 2100

MATLAB:

```
>> A=[1 1; .05 .08]
A =
    1.0000    1.0000
    0.0500    0.0800

>> b=[30000; 2100]
b =
       30000
        2100
>> s=A\b
s =
       10000
       20000
```

Example 4. Saman has 30 coins, consisting of quarters and dimes, which total INR 5.70. How many of each does she have?

Solution Pointers:

Let *x* be the number of quarters. Let *y* be the number of dimes.

Therefore, x + y = 30, and .25x +.10y = 5.7

MATLAB:

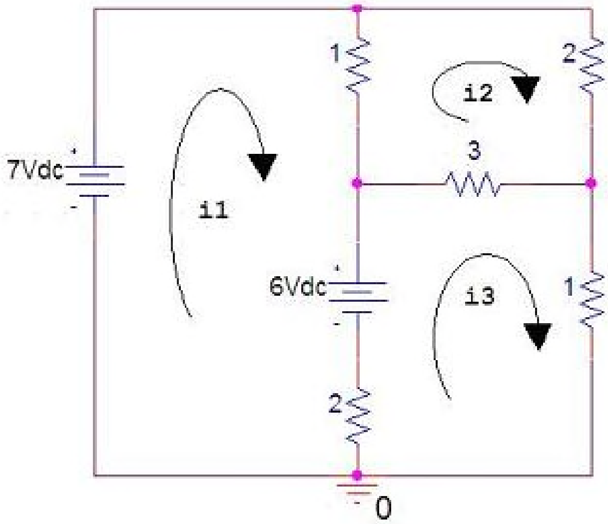
```
>> A=[1 1; .25 .1]
A =
    1.0000    1.0000
    0.2500    0.1000

>> b=[30; 5.7]
b =
   30.0000
    5.7000
```

```
>> s=A\b
s =
    18
    12
```

Therefore, 18 quarters and 12 dimes.

Example 5: Electric Circuit
Consider the following electrical circuit (resistors are in ohms, currents in amperes, and voltages are in volts):



Given the relationship, voltage=resistance multiplied by current, and the circuits shown above, find the values of the currents i1, i2, i3

7 - 1(i1 - i2) - 6 - 2(i1 - i3) = 0
-1(i2 - i1) - 2(i2) - 3(i2 - i3) = 0
6 - 3(i3 - i2) - 1(i3) - 2(i3 - i1) = 0

Simplifying,

$-3i1 + i2 + 2i3 = -1$
$i1 - 6i2 + 3i3 = 0$
$2i1 + 3i2 - 6i3 = -6$

This system can be described with matrices in the form ***Ax*** = ***b***, where ***A*** is the matrix of the coefficients of the currents, ***x*** is the vector of unknown currents, and ***b*** is the column vector of constants on the right of the equalities.

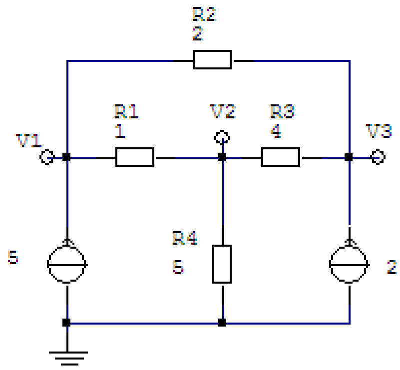
MATLAB:

```
>> A=[-3 1 2 1 -6 3 2 3 -6];
```

```
>> b=[-1 0 -6]'; % where b is the column vector of constants on the right
>> i=A\b
i =
   3.0000
   2.0000
   3.0000
% Therefore,  i1=3; i2=2; i3=3 amperes
```

Example 6: Nodal Analysis

For the circuit shown below, resistors are in ohms and current sources are in amps. F[ind the nodal voltages V_1, V_2, and V_3.



A nodal analysis can be performed by examining each node in a circuit. The goal is to find out what the voltages are in each node with respect to our reference node. We need to know the currents flowing in the circuit and the resistances between each nodes. This is just an application of the Ohm's Law.

Kirchhoff's current law (KCL) states that for any electrical circuit, the algebraic sum of all the currents at any node in the circuit is zero.

In this type of analysis, if there are n nodes in a circuit, and we select a reference node (node 0), the other nodes can be numbered from V_1 through V_{n-1}.
With one node selected as node 0 (reference), there will be n-1 independent equations. If we assume that the admittance between nodes i and j is given as Y_{ij}, we can write the following equations including all of the nodes in the circuit:

where:
$m = n - 1$
$V_1, V_2, ..., V_m$ are voltages from all the nodes with respect to node 0
 is the algebraic sum of current sources at node m.

The above system of equations can be expressed in matrix form as:
$Y V = I$
$V = Y^{-1} I$

Using KCL, and forming our matrices *Y* and *I*, let’s see...
For node V_1 we have,
and the first row of our *Y* matrix is going to be the coefficients for the voltages. This means that we can form our matrix in Matlab like this:
Y(1,:) = [(1/1 + 1/2) -1/1 -1/2];
and naturally I(1) = 5;
At node V_2,
the second row of our Y matrix is going to be the new set of coefficients for the voltages. Letting Matlab work out the operations, we express
Y(2,:) = [1/1 (-1/1 - 1/4 - 1/5) 1/4];
and I(2) = 0;
Finally, at node V_3 we have,
thus, our third row of the Y matrix is
Y(3,:) = [-1/2 -1/4 (1/2 + 1/4)];
and I(3) = 2;

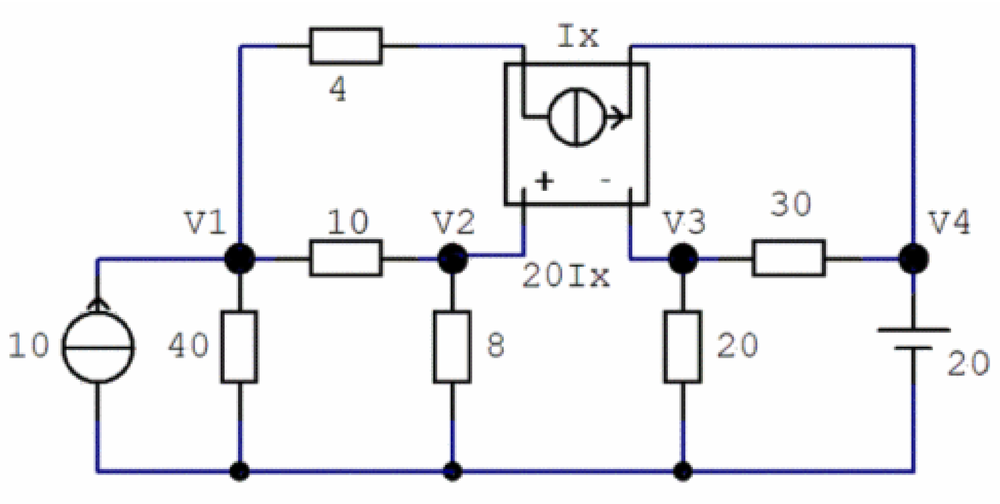
MATLAB

```
>> Y(1,:)=[(1/1+1/2) -1/1 -1/2];
>> Y(2,:)=[1/1 (-1/1 - 1/4 -1/5) 1/4];
>> Y(3,:)=[-1/2 -1/4 (1/2+1/4)];
>> Y
Y =
    1.5000   -1.0000   -0.5000
    1.0000   -1.4500    0.2500
   -0.5000   -0.2500    0.7500
>> I=[5 0 2]';
>> I
I =
     5
     0
     2
>> V=Y\I
V =
   40.4286
   35.0000
   41.2857
```

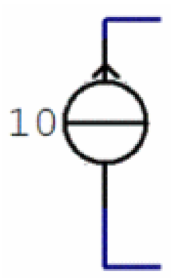
Example 7: Circuit Analysis, building on the previous example. Calculate the values of voltages on the shown four nodes, V1 to V4.

This circuit contains more components: 6 resistors (in ohms), a current source (in amperes), a voltage source (in volts) and a voltage source that produces its voltage depending on a current going through another branch of the circuit (that’s called a current- branch of the circuit (that’s called a current-controlled voltage source, or CCVS).

The schematic for this example is shown below:

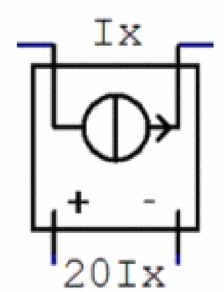


The symbol of the constant current source is:



In particular, the current through this source is 10 A.

The symbol for the current-controlled voltage source is:



In particular, this source is creating a voltage of 20 times the current Ix, that goes through the top branch of the circuit, that is, through the 4-ohm resistor.

We are going to solve a linear system that represents our circuit. We have 4 variables and can study 4 nodes (the remaining node is the reference node or ground), and we are going to solve the linear equation YV = I by using the left division available in Matlab, V = Y\I, where Y is the square matrix (4 × 4) of coefficients of the unknowns in our system, V is the 4-element column vector representing the nodes, and I is the 4-element column vector of constants on the right of the system. Each row of our matrices is going to summarize the analysis of each node, while each column is going to contain the coefficients of the unknowns.

Kirchhoff's current law (KCL) states that for any electrical circuit, the algebraic sum of all the currents at any node in the circuit equals zero.

Analysis

At node 1, we have
Current going to it, 10 A.
Currents going out from it, V1/40, (V1 - V2)/10, and (V1 - V4)/4

In mathematics that means:
10 = V1/40 + (V1 - V2)/10 + (V1 - V4)/4

Rearranging the above numbers, the first row of our Y matrix is:
Y(1, :) = [(1/40 + 1/4 + 1/10) (-1/10) 0 -1/4]
And the first element of our column I is:
I(1) = 10

At nodes 2 and 3, we can see that
V2 – V3 = 20Ix, but we also know that Ix = (V1 - V4)/4
that means that
V2 – V3 = 20(V1 - V4)/4

and we conclude that the second row of our Y matrix is:
Y(2, :) = [5 -1 1 -5]
The second element of our column I is
I(2) = 0

From super nodes 2 and 3, we get
(V2 – V1)/10 + V2/8 + V3/20 + (V3 - V4)/30 = 0

This produces our third row of the Y matrix:
Y(3, :) = [-1/10 (1/10 + 1/8) (1/20 + 1/30) -1/30]

The third element of I is
I(3) = 0

And at node 4, we easily see that V4 = 20.
This means that
Y(4, :) = [0 0 0 1]
and
I(4) = 20

Because of the above explanation, our Y matrix is
Y = [(1/40 + 1/4 + 1/10) -1/10 0 -1/4
5 -1 1 -5

-1/10 (1/10 + 1/8) 1/20 + 1/30 -1/30
0 0 0 1]

Our constant vector on the right of our system is
I = [5 0 0 20]'

And, thanks to the left division, we can easily solve our simple circuit, by just typing
V = Y\I

Nodal voltages V1, V2, V3, V4 are

```
V  =
   36.2215
   35.8306
 - 45.2769
   20
```

MATLAB

```
>> Y(1,:)=[(1/40+1/4+1/10) (-1/10) 0 -1/4];
>> Y(2,:)=[5 -1 1 -5];
>> Y(3,:)=[-1/10 (1/10+1/8) (1/20+1/30) -1/30];
>> Y(4,:)=[0 0 0 1];
Y =
    0.3750   -0.1000         0   -0.2500
    5.0000   -1.0000    1.0000   -5.0000
   -0.1000    0.2250    0.0833   -0.0333
         0         0         0    1.0000

>> I=[5 0 0 20]'
I =
     5
     0
     0
    20
>> V=Y\I
V =
   36.2215
   35.8306
  -45.2769
   20.0000
```

Some more simple examples

The examples that follow purposely deal with simple real world scenarios that could be viewed analytically and modeled mathematically. The analytical thinking cultivated through simple examples will naturally extend to an ability for solving complex through the process of parallel or lateral thinking.

Example 9: The age three persons x, y, z are represented by the following simultaneous equation. Find the ages.

x+y+z=120; y=x-z; and x=3z
Therefore the equations are:
x+y+z=120; -x+y+z=0; x-3z=0

In MATLAB:

```
>> A=[1 1 1;-1 1 1;1 0 -3];
>> B=[120;0;0];
>> S=inv(A)*B
S =
      60.00
      40.00
      20.00
```

Example 10: There are 24 coins that are worth $4.5. How many are quarters? How many are dimes?
Simultaneous equations:
q+d=24; 25q+10d=450

MATLAB:

```
>> A=[1 1;25 10];
>> B=[24; 450];
>> d=det(A)
d =
     -15.00
>> S=inv(A)*B
S =
      14.00
      10.00
```

Example 11: A professor has fixed three mid-term exams for the same day. CS499 is really hard, requiring twice as much prep time as the other two exams. Of the other two CS200 and CS250, you want to spend two more opf prep time on CS250. Write Systems of linear equations and solve the problem allocating prep time, allowing for 8 hours of sleep.
Solution: Let thre three courses be denoted by x,y, and z.
x+y+z=16; x=2*(y+z); z=y+2; z=y+2;
Therefore,
x+y+z=16; x-2y-2z=0; and -y+z=2.

MATLAB:

```
>> A=[1 1 1;1 -2 -2;0 -1 1];
```

```
>> B=[16;0;2];
>> S=inv(A)*B
S =
  10.6667
   1.6667
   3.6667
```

Example 12 : A dietitian wishes to plan a meal around three foods. The meal is to include 8800 units of vitamin A, 3380 units of vitamin C, and 1020 units of calcium. The number of units of the vitamins and calcium in each ounce of the foods is summarized in the following table:

	Food I	Food II	Food III
Vitamin A	200	500	800
Vitamin C	110	570	340
Calcium	90	30	60

Determine the amount of each food the dietitian should include in the meal in order to meet the vitamin and calcium requirements.
The simultaneous equations are:
200x+500y+800z=2800; 110x+570y+340z=1380; and 90x+30y+60z=200

Diet Planning

```
A=[200 500 800;110 570 340;90 30 60];
B=[2800; 1380; 200];
>> D=det(A)
D =
  -21600000.00
>> S=inv(A)*B
S =
        -0.08
         0.54
         3.18
```


Sources:
http://www.algebra.com/algebra/about/history
http://www.tau.ac.il/~corry/publications/articles/pdf/algebra%20EB.pdf
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