Curve Fitting Story

Curve fitting is the process of constructing a curve, or mathematical function, that has the best fit to a series of data points, possibly subject to constraints. Curve fitting can involve either interpolation, where an exact fit to the data is required, or smoothing, in which a "smooth" function is constructed that approximately fits the data. A related topic is regression analysis, which focuses more on questions of statistical inference such as how much uncertainty is present in a curve that is fit to data observed with random errors. Fitted curves can be used as an aid for data visualization, to infer values of a function where no data are available, and to summarize the relationships among two or more variables. Extrapolation refers to the use of a fitted curve beyond the range of the observed data, and is subject to a degree of uncertainty since it may reflect the method used to construct the curve as much as it reflects the observed data.

Curve fitting is a powerful and widely used analysis tools. Curve fitting examines the relationship, f (x, y) between one or more predictors (independent variables x) and a response variable (dependent variable y), with the goal of defining a "best fit" model of the relationship.

Imagine having to describe the results of an experiment by showing pages after pages of raw and derived data. Not only would a careful study of the data be tedious and unlikely to be used by any but the most dedicated, the data trends would be most difficult to discern. It may not be sufficient to know that as the data for the controlled variable increases, so does data for the output variable. How much of an increase is important? What is the shape of the increase? A mathematical expression can tell this at a glance.

Mathematical equations remove undesired variations from data. Sources of these variations (often called ‘noise’ or ‘artifacts’) can range from strictly random thermodynamic events to systematic effects of electrical power-line magnetic and electrical fields. Everywhere in the building where I work, signals from a strong local radio station appear on sensitive electronic equipment. This adds considerably to the noise seen on our results, and strengthens the advantages of curve-fitting.

Mathematical equations can be used to derive theoretical implications concerning the underlying principles relating variables to one another. Biologists often express these relationships in exponential terms. Sometimes they use hyperbolas. Each of these carries with it a fundamental notion concerning the connection between one variable and another. There is confidence that the relationship means more than just a blind attempt at describing data, and that interpolated or extrapolated values can be obtained without the expectation of too much error.

This brings us to two very important uses of the fitted curve. Interpolation is the process of obtaining a result which would likely have been obtained if the input variable would have been held at some particular value.

Curve fitting differs from the statistical process of regression in that the latter is often the most rational way of achieving the former. In curve fitting, a greater emphasis is placed on the form of the curve which is to be used to match the data, whereas regression often is applied without much thought given to curve selection. In some respects, then, both processes are complementary.

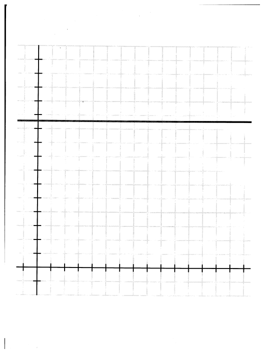
Curve fitting can be viewed as an overall process for which statistical procedures can sometimes be helpful. Often these procedures are blindly used without thorough investigation of the data, experimental procedures used, and inferences to be drawn. When this happens, statistical procedures can actually lead to misleading results.

One cannot overstress the importance of plotting the data before proceeding. One must see what the data look like before a decision is made to treat the data one way or another.

In search of relationships

1. Independence (Non-Relationship)

Take a look at the curve shown below. No matter what value the x variable takes on the curve, the y variable stays the same.



This is a classic example of a relationship called independence. Two quantities are independent if one has no effect on the other. The curve is a horizontal, straight line represented by the general form equation:
y = k where k is a constant.

A suitable conclusion statement from such a relationship would be that: y is independent of x; y does not depend on x. y is constant for all values of x; y is not affected by x; y and x are independent. There is no need to look for a fit between x and y because y values do not depend on x values.

For example,

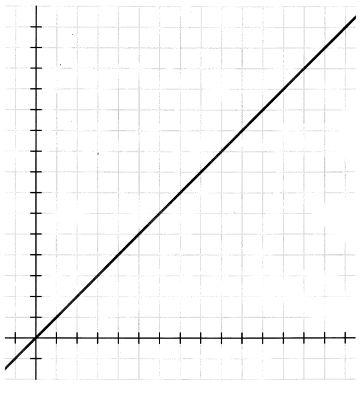
- Free fall acceleration is independent of mass. Heavy objects fall just as fast as light objects in the absence of air resistance.
- The period of a simple pendulum does not depend on its mass. Simple pendulums that are identical in all respects except for the weight of the mass on the end will swing back and forth in an identical manner.
- The speed of light in a vacuum c is constant for all values of v, the speed of the reference frame. No matter how I move around, the speed of light in a vacuum always stays the same.
- The force of dry friction is not affected by the area of the two surfaces in contact. Dragging a box on its bottom or its side results in the same friction force.
- Mass and location are independent. If a frozen turkey has a mass of 10 kg in New York it'll have a mass of 10 kg

in New Jersey, in New Delhi, on Mount Everest, in an airplane, in orbit, on the surface of the moon, in the Andromeda Galaxy, in…. Well, you get the idea.

Independence relationships can be both boring and profound. Boring when we realize there's no link between the two quantities. Profound when we realize we've identified a fundamental principle or underlying concept of great significance. The independence of the speed of light and the speed of a reference frame is one of these statements. The speed of light is a fundamental constant — one of three or four in physics.
direct

2. Straight Dependence Relationship

Now take a look at this curve.



As the x variable increases, the y variable increases too. But there are a lot of curves that do this. What makes this one unique? What distinguishes it from all the other curves that increase (as the mathematicians say) monotonically? The key is in the shape — a straight, non-horizontal line that runs through the origin. With this particular shape, something special happens.

Pick a point on the line and note its coordinates. Double the value of the x variable and see how the y variable responds. The new value of y should also have doubled. Try it again. Only this time, cut the x variable in half. The y variable should have responded in the same manner; that is, it too should be cut in half. Whatever x does, y does the same. This illustrates the simplest, nontrivial form of proportionality — direct proportionality. Two quantities are directly proportional if their ratio is a constant.

$y/x=k$, rearranging this definition gives us the general form equation, $y = kx$, where k is the constant of proportionality, which everyone should recognize as the slope of a straight line in the x,y plane.

A suitable conclusion statement from such a relationship would be that
y is directly proportional to x; y varies directly with x; y and x are directly proportional; $y \propto x$

For example

- Regular wages are directly proportional to the number of hours worked. Forty hours of work pays four times as much as ten hours of work. One hour of work pays one-tenth as much as ten hours of work.
- Weight varies directly with mass. Three times more mass means three times more weight, too. Likewise, half the mass means half the weight.
- Distance and time are directly proportional when speed is constant. Driving for two hours gets you twice as far away as one hour would, but only half as far as four hours.
- Warning! Don't think that directly proportional means "when one increases, the other increases" or "when one decreases, the other decreases". It's a more specific kind of relationship than that. Here's a contrary example. A worker who puts in 60 hours on the job works 1.5 times as much as one who puts in 40 hours.

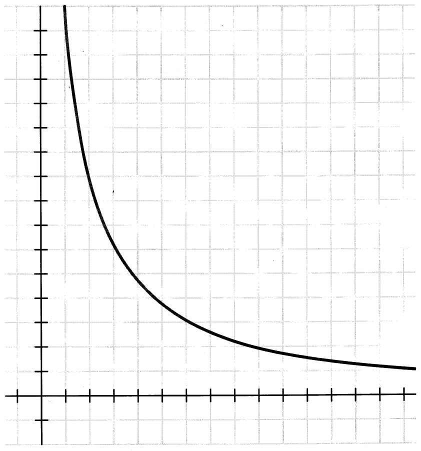
60 hr/40 hr=1.5

But workers working making more than 40 hours a week in the US are supposed to be paid at an overtime rate, which is

typically one and a half times their regular wage. Thus the 60 hour-a-week worker earns 1.75 times as much as the 40 hour-a-week worker.
(1x40 regular hours + 1.5x20 overtime hours)/(1*40 regular hours)=1.75

Since the changes are not the same, $1.75 \neq 1.5$, the wages earned in this example are not directly proportional to the hours worked. A direct relationship is much more special than the general statement, "when one increases, the other increases". It's more like, "when one changes by a certain ratio, the other changes by the same ratio".

3. Inverse Relationship



Moving on. Take a look at this curve. This shape is called a rectangular hyperbola — a hyperbola since it has asymptotes (lines that the curve approaches, but never crosses) and rectangular since the asymptotes are the x and y axes (which are at right angles to one another).

Some say that this curve shows the opposite behavior of the previous one; that is, as the x variable increases, the y variable decreases and as the x variable decreases, the y variable increases. But like the previous curve there's a more specific kind of change that takes place. Check it out for yourself. Pick a convenient point on the curve. Note the coordinate values at this point. Now double the x coordinate and see what happens to the y coordinate. It's cut in half. Now try the reverse. Pick a point on the curve and cut its x coordinate in half. The y coordinate is now double its original value. Triple x and you get one-third of y. Reduce x to one-fourth and watch y increase by four. However you change one of the variables the other changes by the inverse amount. This illustrates another simple kind of proportionality — inverse proportionality. Two quantities are said to be inversely proportional if their product is a constant.

$X*y = k$

Rearranging this definition gives us the general form equation:

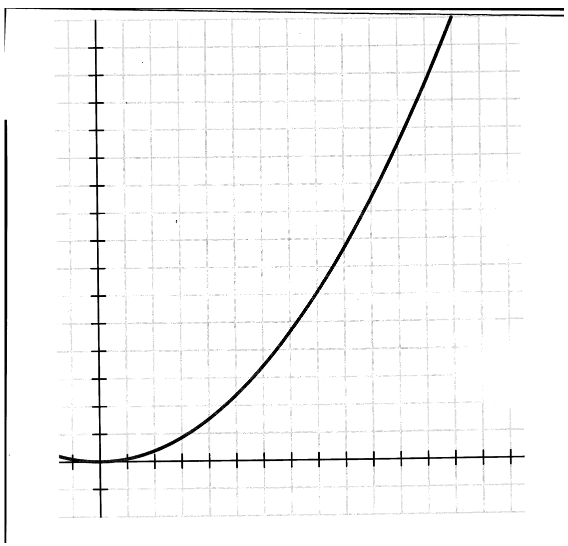
$y=k/x$

where k is the constant of proportionality.

A suitable conclusion statement from such a relationship would be that:
y is inversely proportional to x; y varies inversely as x; y and x are inversely proportional; $y \propto 1/x$ or $y \propto x{-1}$.

For example:
- The time needed to finish a job varies inversely as the number of workers. More workers means less time to finish a job. (Twice as many means it takes half the time.) Fewer workers means it takes longer. (If only one-third of the normal number of workers show up, the job will take three times longer.)
- The volume of a mass of gas is inversely proportional to the pressure acting on it. Place a balloon in a hyperbaric chamber and double the pressure — the balloon will squash to half its original volume. Place the balloon in a vacuum chamber and decrease the pressure to one-tenth atmospheric — the balloon will expand ten times in volume (assuming it doesn't break first).

4. Square Relationship



What do we have here? Why it's a parabola with its vertex at the origin. You get this kind of curve when one quantity is proportional to the square of the other. Since this parabola is symmetric about the y-axis that makes it a vertical parabola and we know that it's the horizontal variable that gets the square. Here's the general form equation for this kind of curve:

$y = kx^2$

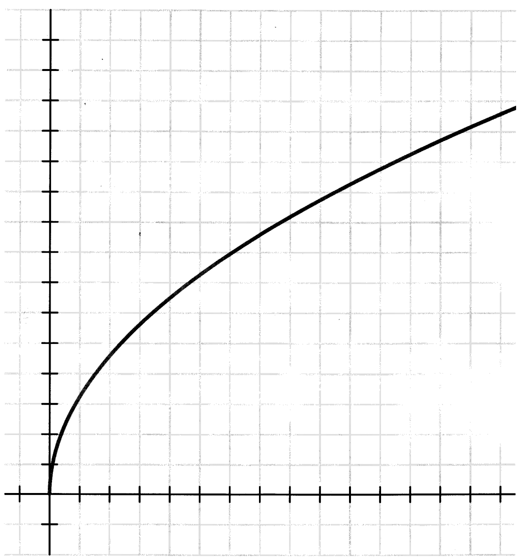
A suitable conclusion statement from such a relationship would be that:

y is proportional to the square of x; $y \propto x^2$;

For example:

- The distance traveled by an object dropped from rest is proportional to the square of time. How long does it take to fall one meter? Double that time and you'll fall 4 m, triple it and you'll fall 9 m, and so on.
- The rate at which heat is produced by an electric circuit is proportional to the square of the current. Doubling the current in a toaster oven quadruples its heat output. Reduce the current in the CPU of a computer to half its previous value and you'll reduce the heat output to one-quarter its previous value.

5. Square Root Relationship



Here's another parabola with its vertex at the origin. This one's tipped on its side and is symmetric about the x-axis. For a horizontal parabola like this one, it's the vertical variable that gets the square. The general form equation for this kind of curve is:

$y = k\sqrt{x}$

A suitable conclusion statement from such a relationship would be that:
y is proportional to the square root of x; $y \propto \sqrt{x}$ or $y \propto x^{1/2}$

For example:

- Speed is proportional to the square root of distance for freely falling objects. How fast is an object moving after it has fallen one meter? At four meters it'll have double that speed; at nine meters, triple; sixteen, quadruple; and so on.

Something to remember — the square root is not an explicit function. It isn't single-valued. Every number has two square roots: one positive and one negative. Typical curve fitting software disregards the negative root, which is why I only drew half a parabola on the diagram above. Something else to remember — the domain of the square root is restricted to non-negative values. That's a fancy way of saying you can't find the square root of a negative number (not without expanding your concept of "number", that is).

6. Power Relationship

So far, we have five curves and five general form equations:

• independent	$y = k$
• direct	$y = kx$
• inverse	$y = k/x$
• square	$y = kx^2$
• square root	$y = k\sqrt{x}$

They have three common components.

x =	an independent variable (or explanatory variable)
y =	a dependent variable (or response variable)
k =	a constant of proportionality

and one component that varies:

n =	power of the independent variable

We could rewrite these general equations with two variables, a constant of proportionality and a power like this…

• independent	$y = kx^0$
• direct	$y = kx^1$
• inverse	$y = kx^{-1}$
• square	$y = kx^2$
• square root	$y = kx^{½}$

We could even go so far as to write a general form equation for a whole family of equations:

$$y = kx^n$$

Any two variables that are related to one another by an equation of this form are said to have a power relation between them.

power	general form	description	appearance
0	$y = k$	independent	horizontal, straight line
1	$y = kx$	direct	non-horizontal straight line through the origin
2	$y = kx^2$	square	vertical parabola with vertex at the origin
3	$y = kx^3$	cube	

−1	$y = k/x$	inverse	rectangular hyperbola
−2	$y = k/x^2$	inverse square	
−3	$y = k/x^3$	inverse cube	
½	$y = k\sqrt{x}$	square root	horizontal parabola with vertex at the origin
⅓	$y = k\sqrt[3]{x}$	cube root	

Power relationships summarized

7. Simple Linear Relationship

Description: A combination of constant and direct. A fixed amount is added (or subtracted) at regular intervals.
General form.
$y = ax + b$

A suitable conclusion statement from such a relationship would be that:
y is linear with x; y varies linearly with x; y is a linear function of x;

Appearance: any straight line, regardless of slope or y-intercept
Example(s): utility bills (there's always a service charge)

Engineering Applications of Simple Linear Regression (Curve Fitting)

The least-squares technique for finding a linear regression of the form $y = ax + b$ is critical in engineering, as all sampled data always has an error associated with it, and while models may suggest that the response of a system should be linear, the actual output may less obviously be so, for any number of reasons, including limitations in measuring equipment, random errors and fluctuations, and unaccounted variables.

The method of least squares results in a fast and efficient manner of finding the best fitting straight line which may pass through given data, yielding approximations of unknown coefficients.

For example, it is well known that an *ideal* resistor is linear in its response. This, however, may be less true in practice. Simply using one reading with one current to approximate the resistance of a resistor has two problems:

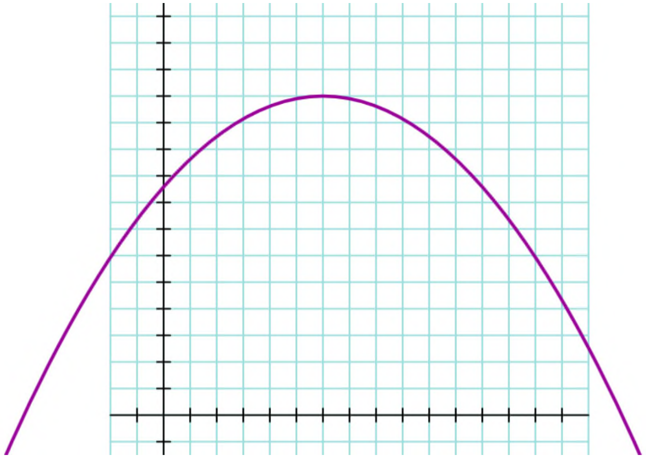
- We cannot give any estimate as to what the error of our approximation is, and
- The resistor may not be linear for the given range of currents, hence our approximation may be completely inaccurate because the model of an ideal linear resistor does not even apply.

Using multiple readings and linear regression gives us the ability to make much more definitive statements about the accuracy of our approximation and the applicability of our model.

8. Quadratic Relationship

Description: A combination of square, direct, and constant.

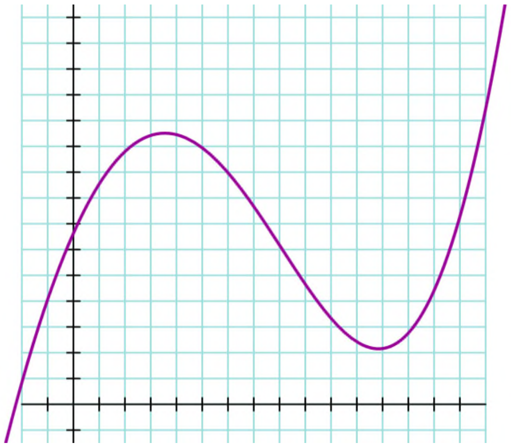
General Form
$y = ax^2 + bx + c$
A suitable conclusion statement from such a relationship would be that:
y is quadratic with x; y varies quadratically with x; y is a quadratic function of x.



Appearance: A vertical parabola when graphed. It's vertex can be anywhere. It could also be flipped upside down.
Example(s): distance during uniform acceleration

9. Polynomial Relationship

Description: A combination of a constant, direct, square, cube, …. Keep going as far as you wish.
General form.
$y = a + bx + cx^2 + dx^3 + \ldots$



A suitable conclusion statement from such a relationship would be that:

- y can be approximated by an nth order polynomial of x.
- An nth order polynomial of x was fit to y.

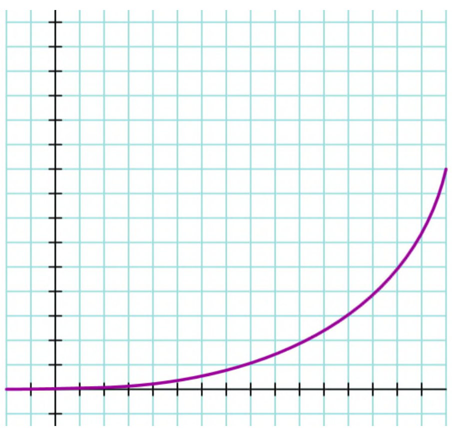
Appearance: any non-periodic function without asymptotes
Example(s): Polynomial functions can be used to approximate many continuous, single-valued curves

order	general form	name
0	$y = a$	constant
1	$y = a + bx$	linear
2	$y = a + bx + cx^2$	quadratic
3	$y = a + bx + cx^2 + dx^3$	cubic

4	$y = a + bx + cx^2 + dx^3 + ex^4$	quartic
5	$y = a + bx + cx^2 + dx^3 + ex^4 + fx^5$	quintic
⋮	⋮	⋮
n	$y = a_0x^0 + a_1x^1 + a_2x^2 + a_3x^3 + \ldots = \sum_{i=1}^{n} aixi$	nth order polynomial

Polynomial relationships summarized

10. Exponential Growth Relationship



Description:
General form.
$y = an^{bx}$

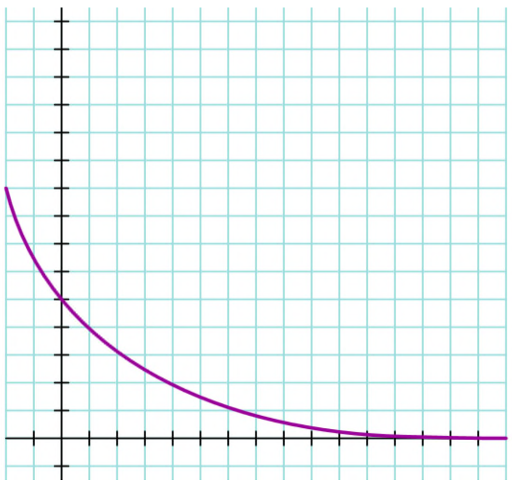
A suitable conclusion statement from such a relationship would be that:
y increases exponentially with x; y grows exponentially with x; $y \propto n^x$

The ratio of successive iterations is a constant. The quantity is multiplied by a fixed amount at regular intervals.

Appearance: asymptotic with negative x-axis, followed by runaway expansion

Example(s): unrestricted population growth, the magic of compound interest

11. Exponential Decay

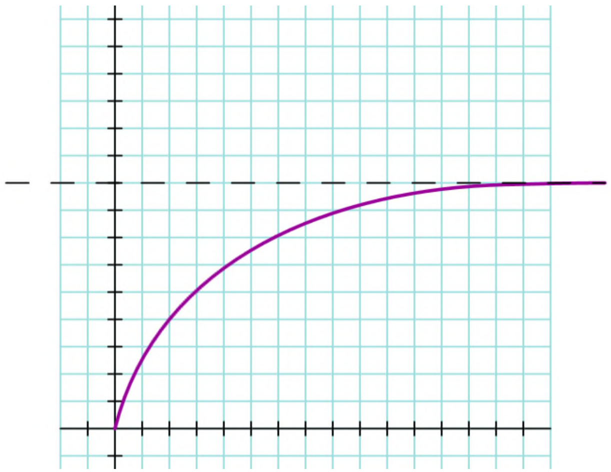


Description:
General form.
$y = an^{-bx}$

A suitable conclusion statement from such a relationship would be that:
y decreases exponentially with x; y decays exponentially with x; y ∝ n−x

The ratio of successive iterations is a constant. The quantity is divided by a fixed amount at regular intervals.
Appearance: large initial value followed by abrupt collapse, approaches positive x-axis asymptotically
Example(s): radioactive decay, discharging a capacitor, de-energizing an inductor

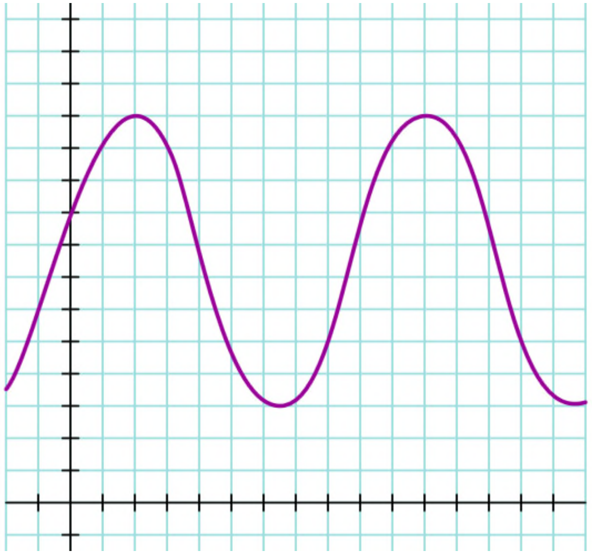
12. Exponential Approach Relationship



Description:
General form.
$y = a\,(1 - n^{-bx}) + c$
A suitable conclusion statement from such a relationship would be that…
y approaches a final value exponentially.

Appearance: asymptotically approaches a horizontal line
Example(s): charging a capacitor, energizing an inductor, teaching (half the students get it, then half of the remaining students get it, then half of the remaining students get it, and so on…)

13. Periodic Type Relationship



Description:
General form.
y = a sin (bx + c)
A suitable conclusion statement from such a relationship would be that:
y varies periodically with x; y is periodic with x.

Appearance: A sine curve is the prototypical example, not the only example. Any curve that repeats itself is periodic.

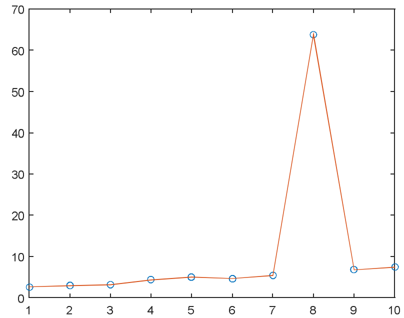
Perceived Relationship and Curve Fitting

The lesson to be drawn from the over a dozen type of relationships, as described above, is that curve fitting is not a blind process. We cannot simply choose any relationship at whim in finding a mathematical fit to observed data. There should some underlying reason for choosing a particular relationship to observed data.

The goal of curve fitting is to find a mathematical relationship that fits experimentally or empirically collected data from observations. Either the scientist or engineer collecting the data has a prior notion of the kind of kind of mathematical relationship that applies to the data, or he can generate a plot from the data to form ideas about the relationship that may exist based on a plot of the data and the knowledge of the types of relationships described above.

Consider the following sample data:
x= [1 2 3 4 5 6 7 8 9 10]
y= [2.62 2.91 3.13 4.29 4.99 4.65 5.40 63.85 6.75 7.39]

The plot of function f (x, y) is shown below:



As one can see, this plot is quite revealing. It shows the shape of a linear relationship which can be approximated by a straight-line mathematical function such as f(x)=ax+b except for one very unusual data value of y at x=8. Such unusual data values are usually caused by errors in observations as will be shown below.

In an effort to find simplest possible relationship, one may start with the mathematical representation of a simple linear relationship:

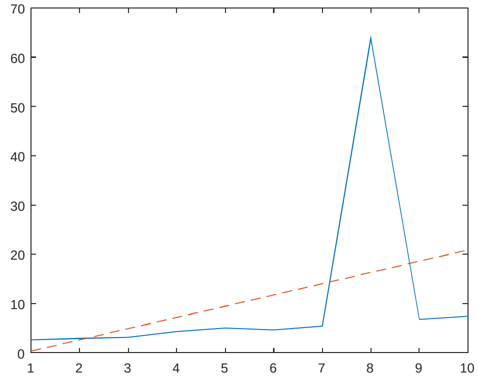
$$f(x,y) = ax+b$$

This simple form of relationship may produce a satisfactory outcome in many cases.

Example 1: Find the least-squares curve which fits the linear data**:**
(1, 2.6228), (2, 2.9125), (3, 3.1390), (4, 4.2952), (5, 4.9918),
(6, 4.6468), (7, 5.4008), (8, 63.853), (9, 6.7494), (10, 7.3864)

In MATLAB:

```
>> simplepolyfit
Type the x values as a vector enclosed within [ ]:
[1 2 3 4 5 6 7 8 9 10]
Type the observed y values as a vector enclosed within [ ]:
[2.62 2.91 3.13 4.29 4.99 4.65 5.40 63.85 6.75 7.39]
coeffs =
   2.2812   -1.9487
```



Note:
The continuous line in blue is drawn through all observed data points. The dashed line in red is the curve (line) derived from the least squares method of minimizing deviations (called regression) from observed data.

The 8th point appears to be significantly different from all other values. It is called an outlier. This would almost certainly appear to be an error in measurement or an error in recording. There are two possible solutions:

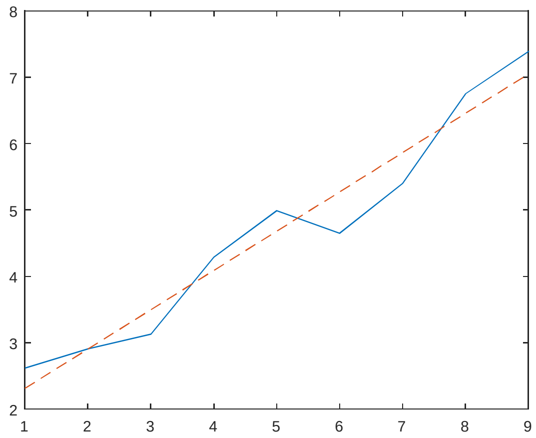
- Leave the point out, or
- Re-sample the value at the point $x = 8$.

If we do nothing, we get the following *best-fitting* line:
$y(x) = 2.28\,x - 1.94$.

If we remove the point as, shown below, and simply use the remaining nine points, we get the line
$y(x) = 0.59\,x + 1.72$.

```
>> simplepolyfit
Type the x values as a vector enclosed within [ ]:
[1 2 3 4 5 6 7 8 9]
```

Type the observed y values as a vector enclosed within [ ]:
[2.62 2.91 3.13 4.29 4.99 4.65 5.40 6.75 7.39]
coeffs =
    0.5917   1.7228



This second curve line (shown in --) fits better with observed data, indicating that not all observed values belong to the population as a whole.

The MATLAB Code for producing the curve fitting outcomes is:

```
% simplepolyfit: Linear curve fitting with polynomial of degree 1
function fitsimple
clear all
x=input ('Type the x values as a vector enclosed within [ ]:\n');
y=input ('Type the observed y values as a vector enclosed within [ ]:\n');
```

```
n=1; % For polynoial of degree 1
% polyfit(x,y,n) returns the coefficients for a polynomial p(x) of degree n that is a best fit
% (in a least-squares sense) for the data in y.
% The coefficients in p are in descending powers, and the length of p is n+1
coeffs=polyfit(x,y,1)
% Curve fitting yc values
% polyval(p,x) returns the value of a polynomial of degree n evaluated at x.
% The input argument p is a vector of length n+1 whose elements are the coefficients
% in descending powers of the polynomial to be evaluated.
yc=polyval(coeffs,x);
plot(x,y,x,yc,'--')
```

Correlation Coefficient

How well does your regression equation truly represent your set of data?

One of the ways to determine the answer to this question is to exam the correlation coefficient, r,, and the coefficient of determination, r^2.

Using the built-in function of MATLAB, the correlation coefficient for two cases above is:
10 data points including the outlier = 0.3678
9 data points excluding the outlier = 0.9732

r=1 for perfect correlation which is rarely the case. r=0.9732 represents a good correlation. r=0.3678 is represents poor correlation which understandable because one case of bad outlier.

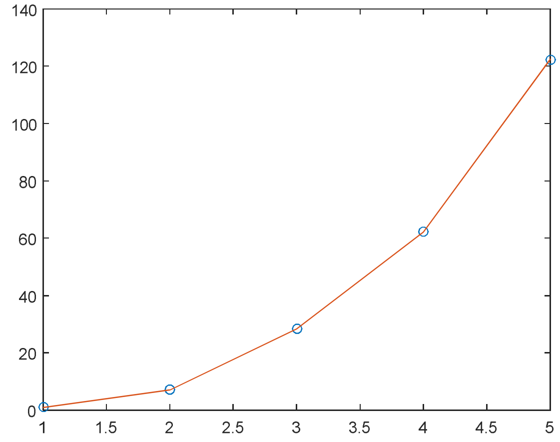
Regression analysis is the study of the relationship between one or several predictors (independent variables) and the response (dependent variable). To perform regression analysis on a dataset, a regression model is first developed.

Then the best fit parameters are estimated using something like the least-square method. Finally, the quality of the model is assessed using one or more hypothesis tests.

From a mathematical point of view, there are two basic types of regression: linear and nonlinear. A model where the fit parameters appear linearly in the Least Squares normal equations is known as a "linear model"; otherwise it is "nonlinear". In many scientific experiments, the regression model has only one or two predictors, and the aim of regression is to fit a curve or a surface to the experimental data. So we may also refer to regression analysis as "curve fitting" or "surface fitting."

Example 2: Nonlinear Mathematical Relationship Curve
Given data:
x = [1 2 3 4 5]
y = [0.9 7.0 28.3 62.1 122.4]

The plot from the is shown below:



What can we surmise from about the possible applicable mathematical relation? A review of the thirteen relationship patterns described in the preceding paragraphs, the likely choice is a cubic power relationship such as:

$f(x) = ax^n$

The curve fitting operation will determine the values of a and n, with some initial guesses such as a=1 and n=3 based on the shape of the plot for the data.

To carry out nonlinear fits, we need the following:

- A function to evaluate the model for a given set of parameters and for a given time (this is the curve we are fitting to the data)
- A function to calculate the sum of the squares of the errors between the model and the data (for a given set of fitting parameters)
- A routine to put everything together

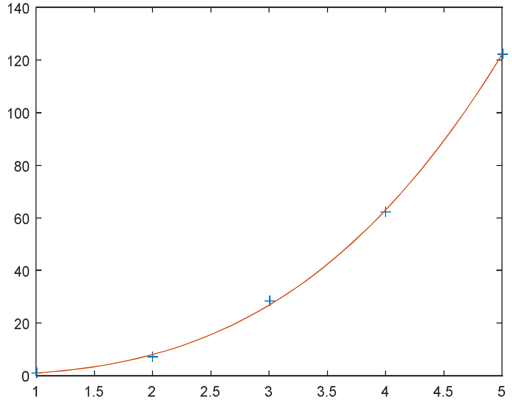
MATLAB provides a built in functions called fminsearch (fun,x0,options) for this purpose. fminsearch finds the minimum of a scalar function of several variables, starting with an initial estimate. This is generally referred to as unconstrained nonlinear optimization.

An example of using this function is:
x = fminsearch(fun,x0,options)
where x is a function handle, fun creates the handle for input x0, and options are used for conveying related information such as the initial guesses for parameters a and n.

The resulting solution is shown below.

In MATLAB

```
>> nonlinfitcubic
Mathemation relation to be used is: ax^n
Type the x values as a vector enclosed within [ ]:
[1 2 3 4 5]
Type the observed y values as a vector enclosed within [ ]:
[0.9 7.0 28.3 62.1 122.4]
Type an initial estimate for parameter a:
1
Type an initial estimate for parameter n:
3
Value of parameter a is 1.026723
Value of parameter n is 2.969526

The r^2 value for this fit is 0.999582
```



The continues line shown in red is the mathematical function and blue markers in blue are the data values of function y.

Therefore, the fitted cubic power relationship is:
$1.026723x^{2.969526}$
More simply,
$1.03x^{2.97}$

The MATLAB code for curve fitting using the given cubic power relationship is:

```
function nonlinfitcubic
clear all
disp('Mathematical relation to be used is: ax^n')
x=input('Type the x values as a vector enclosed within [ ]:\n');
y=input('Type the observed y values as a vector enclosed within [ ]:\n');
a=input('Type an initial estimate for parameter a:\n');
n=input('Type an initial estimate for parameter n:\n');
numpts=max(size(x));
p(1)=a; %guess for first parameter
p(2)=n; %guess for second parameter
zout=fminsearch(@(z) sumoferrs(z,x,y), p);
fprintf('Value of parameter a is %f\n',zout(1))
fprintf('Value of parameter n is %f\n',zout(2))
xplot=x(1):(x(end)-x(1))/(10*numpts):x(end);
yplot=curve(xplot,zout);
plot(x,y,'+',xplot,yplot)
% The following lines attempt to assess the quality of the fit
datamean=mean(y);
errorsum=0;
othersum=0;
```

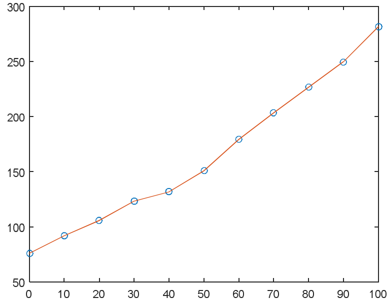
```
for  i=1:numpts
    errorsum=errorsum+(curve(x(i),zout)-y(i))^2;
    othersum=othersum+(datamean-y(i))^2;
end
rsquared=1-errorsum/othersum;
fprintf('The r^2 value for this fit is %f\n',rsquared)

function f=curve(x,z)
a=z(1);
n=z(2);
f=a*x.^n;

function f=sumoferrs(z, x, y)
f=sum((curve(x,z)-y).^2);
```

Example 3: Exponential Decay Curve Fitting

Years 1990-2010 scaled to 0-100

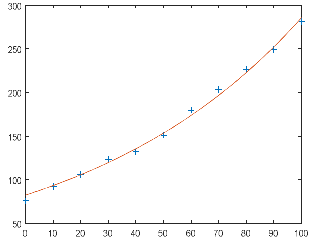
x= [0 10 20 30 40 50 60 70 80 90 100]

y= [75.995 91.972 105.711 123.203 131.669 150.697 179.323 203.212 226.505 249.633 281.422]



This plot provides clues that the population growth is slow. Also, at year 1900 scaled to year 0, the function value is around 75. If we substitute x=0 in the mathematical relation $f(x)=ke^{ax}$, we find y is approximately 75. That is our guess for k. Also, at x=10 with k=75, the value of parameter 'a' has to be just a fraction for the data value on the graph. Thus our guess for a=.1.

In MATLAB:
>> nonlinfitexp
Mathematical relation to be used is: ke^at
Type the x values as a vector enclosed within [ ]:
[0 10 20 30 40 50 60 70 80 90 100]
Type the observed y values as a vector enclosed within [ ]:
[75.995 91.972 105.711 123.203 131.669 150.697 179.323 203.212 226.505 249.633 281.422]
Type an initial estimate for parameter k:
75
Type an initial estimate for parameter a:
.1
Value of parameter k is 82.412865
Value of parameter a is 0.012432
The r^2 value for this fit is 0.995880

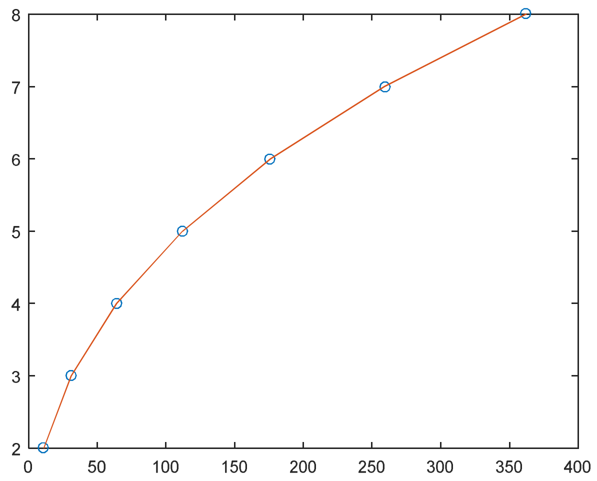


```
function nonlinfitexp
clear all
disp('Mathematical relation to be used is: ke^ax')
x=input('Type the x values as a vector enclosed within [ ]:\n');
y=input('Type the observed y values as a vector enclosed within [ ]:\n');
k=input('Type an initial estimate for parameter k:\n');
a=input('Type an initial estimate for parameter a:\n');
numpts=max(size(x));
p(1)=k; %guess for first parameter
p(2)=a; %guess for second parameter
zout=fminsearch(@(z) sumoferrs(z,x,y), p);
fprintf('Value of parameter a is %f\n',zout(1))
fprintf('Value of parameter n is %f\n',zout(2))
xplot=x(1):(x(end)-x(1))/(10*numpts):x(end);
yplot=curve(xplot,zout);
plot(x,y,'+',xplot,yplot)
% The following lines attempt to assess the quality of the fit
datamean=mean(y);
errorsum=0;
othersum=0;
for  i=1:numpts
    errorsum=errorsum+(curve(x(i),zout)-y(i))^2;
    othersum=othersum+(datamean-y(i))^2;
end
rsquared=1-errorsum/othersum;
fprintf('The r^2 value for this fit is %f\n',rsquared)

function f=curve(x,z)
```

```
a=z(1);
n=z(2);
f=k*exp(a*x);

function f=sumoferrs(z, x, y)
f=sum((curve(x,z)-y).^2);
```

Example 4: For the data shown below, as a result of measurements on the following data was gathered for the applied force and resulting deflection. Find a suitable mathematical relationship fitting this data.

x = [11 31 64 112 176 259 362] the independent data set - force
y = [2 3 4 5 6 7 8] the dependent data set - deflection



Engineering Side Story about Frictional Resistance

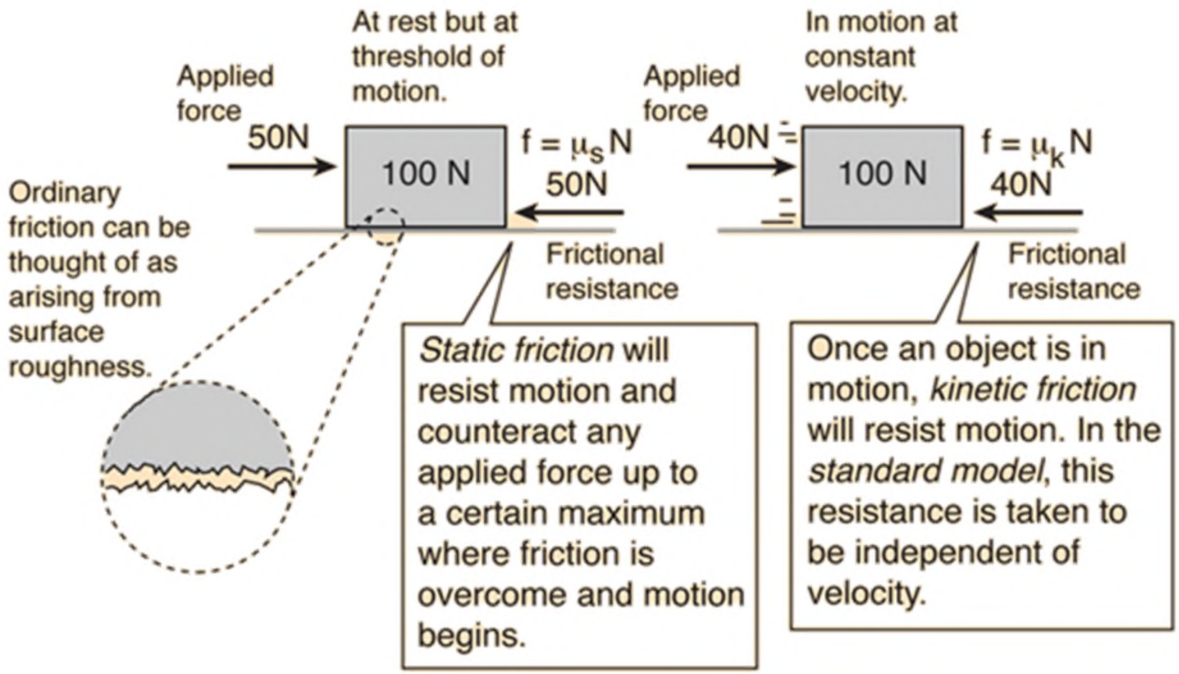
Frictional resistance to the relative motion of two solid objects is usually proportional to the force which presses the surfaces together as well as the roughness of the surfaces. Since it is the force perpendicular or "normal" to the surfaces which affects the frictional resistance, this force is typically called the "normal force" and designated by N. The frictional resistance force may then be written:

$F = \mu N$, where

μ = coefficient of friction which is different when the objects are to move from a stationary state (static friction) than when objects are in motion (kinetic friction). You may view the scenario tire friction when the car starts to move versus when it is in motion.

The frictional force is also presumed to be proportional to the coefficient of friction. However, the amount of force required to move an object starting from rest is usually greater than the force required to keep it moving at constant velocity once it is started. Therefore two coefficients of friction are sometimes quoted for a given pair of surfaces - a coefficient of static friction and a coefficient of kinetic friction. The force expression above can be called the standard model of surface friction and is dependent upon several assumptions about friction.

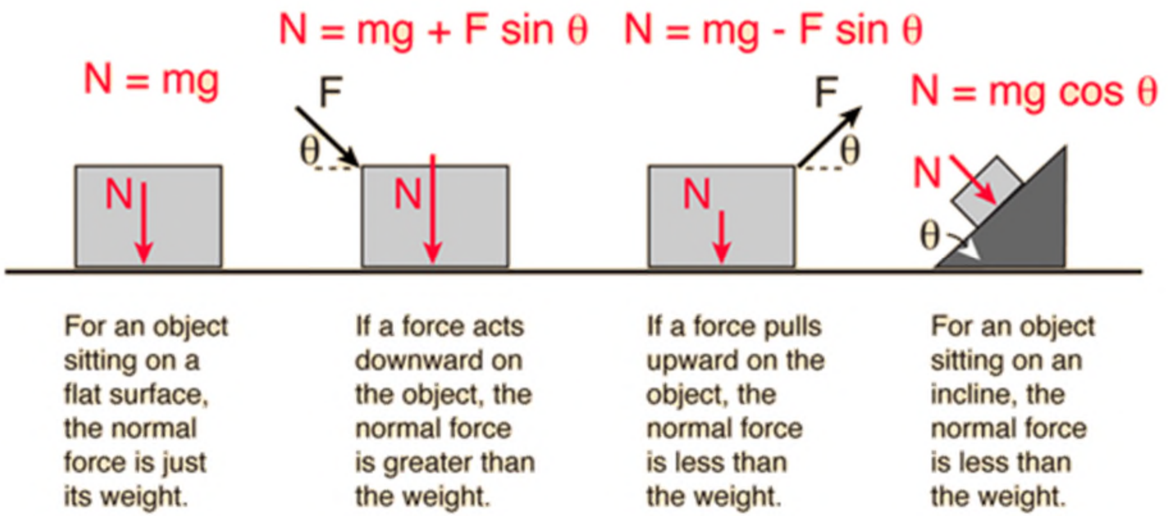
While this general description of friction, referred to as the standard model, has practical utility, it is by no means a precise description of friction. Friction is in fact a very complex phenomenon which cannot be represented by a simple model. Almost every simple statement you make about friction can be countered with specific examples to the contrary. Saying that rougher surfaces experience more friction sounds safe enough - two pieces of coarse sandpaper will obviously be harder to move relative to each other than two pieces of fine sandpaper. But if two pieces of flat metal are made progressively smoother, you will reach a point where the resistance to relative movement increases. If you make them very flat and smooth, and remove all surface contaminants in a vacuum, the smooth flat surfaces will actually adhere to each other, making what is called a "cold weld".



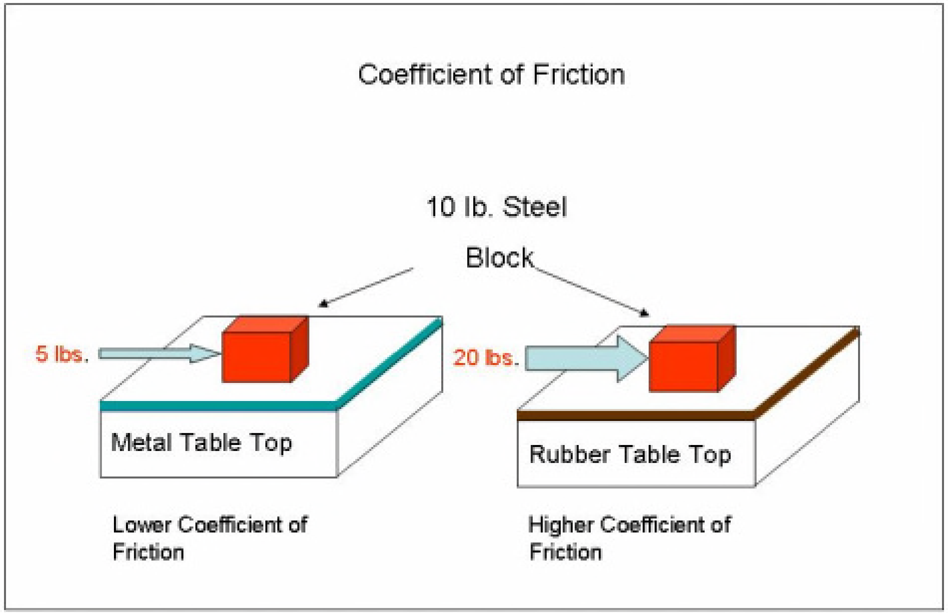
Once you reach a certain degree of mechanical smoothness, the frictional resistance is found to depend on the nature of the molecular forces in the area of contact, so that substances of comparable "smoothness" can have significantly different coefficients of friction.

An easily observed counterexample to the idea that rougher surfaces exhibit more friction is that of ground glass versus smooth glass. Smooth glass plates in contact exhibit much more frictional resistance to relative motion than the rougher ground glass.

Frictional resistance forces are typically proportional to the force which presses the surfaces together. This force which will affect frictional resistance is the component of applied force which acts perpendicular or "normal" to the surfaces which are in contact and is typically referred to as the normal force. In many common situations, the normal force is just the weight of the object which is sitting on some surface, but if an object is on an incline or has components of applied force perpendicular to the surface, then it is not equal to the weight.



The above cases are the commonly encountered situations for objects at rest or in straight line motion. For curved motion, there are cases like a car on a banked curve where the normal force is determined by the dynamics of the situation. In that case, the normal force depends upon the speed of the car as well as the angle of the bank.



Example 5: Given the data as shown below, find f(N,F) corresponding to f(x,y)
x=N= [0 11 21 27 43 52]
y=F= [0 2 4 6 8 10]

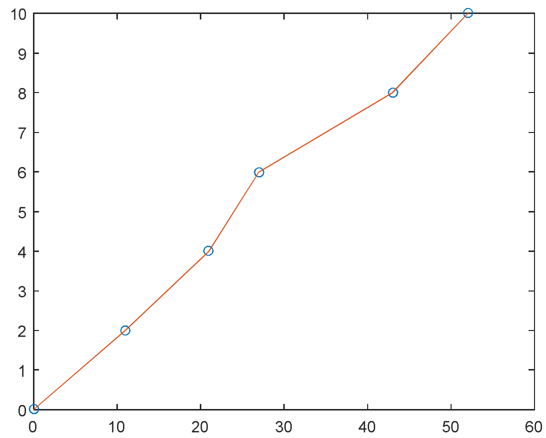
In MATLAB:
>> plotter
Type the x values as a vector enclosed within [ ]:
[0 11 21 27 43 52]
Type the observed y values as a vector enclosed within [ ]:

[0 2 4 6 8 10]

The plotter script is very simple but convenient:

```
% Plotter.m
x=input('Type the x values as a vector enclosed within [ ]:\n');
y=input('Type the observed y values as a vector enclosed within [ ]:\n');
plot(x,y,'o',x,y)
```

Plot of the given (N,F) or (x,y) is shown below:

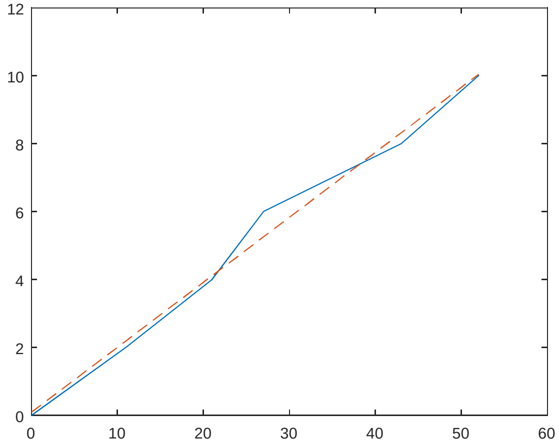


In MATLAB:

```
>> polyfitn
Type the x values as a vector enclosed within [ ]:
[0 11 21 27 43 52]
```

Type the observed y values as a vector enclosed within [ ]:
[0 2 4 6 8 10]
Type the desired polynomial degree:
1
coeffs =
   0.1914   0.0874

The correlation coefficient is:
   0.9949



The plot of the derived curve, $f^o(x,y)$ shows y near zero at x=0, and the coefficient of friction is the slope or gradient value of 0.1914

$y=f^o(x,y)=F= 0.1914R + 0.0874$

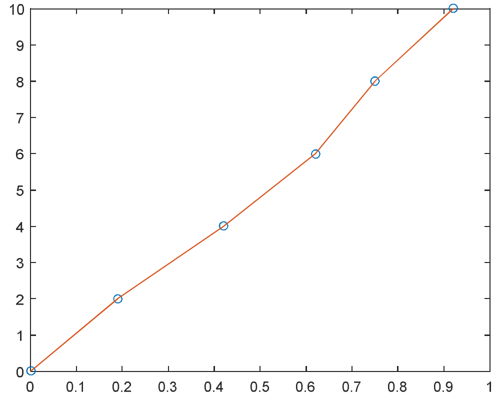
Example 6:
In an experiment to verify Ohm's Law, the voltage and current acting on a resistor was measured and the following results obtained.

V	0	2	4	6	8	10
I	0	0.19	0.42	0.62	0.75	0.92

Assuming the law should be linear, find the best fit straight-line law and check the answers with using Excel. (Calculation gives m (gradient, slope) =10.73, C (intercept) = 0.184 Excel gives same answers)

x= [0 .19 .42 .62 .75 .92]
y= [0 2 4 6 8 10]

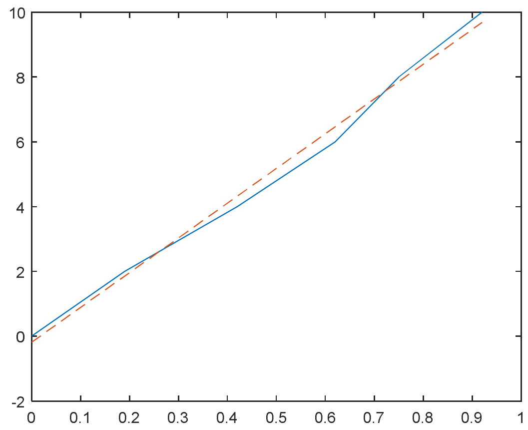
The plot of data is:



derived gradient=10.73, intercept=.184

In MATLAB:

```
>> polyfitn
Type the x values as a vector enclosed within [ ]:
[0 .19 .42 .62 .75 .92]
Type the observed y values as a vector enclosed within [ ]:
[0 2 4 6 8 10]
Type the desired polynomial degree:
1
coeffs =
   10.7261   -0.1843

The correlation coefficient is:
    0.9965
```



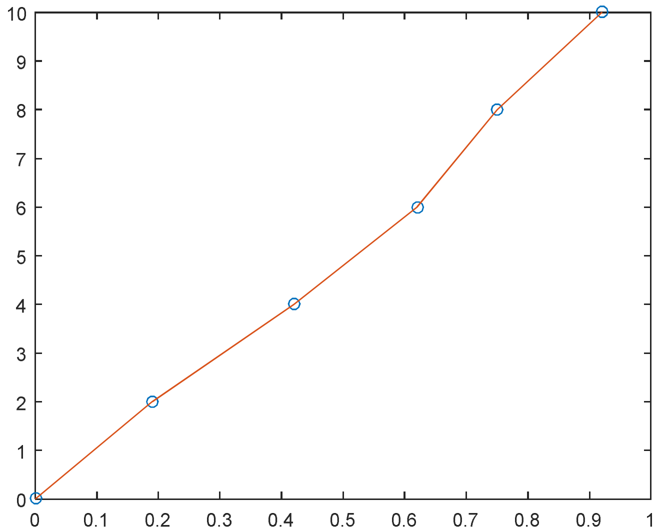
The script for an degree polynomial fit is:

```
% simplepolyfit: Linear curve fitting with polynomial of degree 1
function fitsimple
clear all
x=input('Type the x values as a vector enclosed within [ ]:\n');
y=input('Type the observed y values as a vector enclosed within [ ]:\n');
n=input('Type the desired polynomial degree:\n');
% polyfit(x,y,n) returns the coefficients for a polynomial p(x) of degree n that is a best fit
% (in a least-squares sense) for the data in y.
% The coefficients in p are in descending powers, and the length of p is n+1
coeffs=polyfit(x,y,n)
% Curve fitting yc values
% polyval(p,x) returns the value of a polynomial of degree n evaluated at x.
% The input argument p is a vector of length n+1 whose elements are the coefficients
% in descending powers of the polynomial to be evaluated.
yc=polyval(coeffs,x);
plot(x,y,x,yc,'--')
% Correlation coefficient r
r=corrcoef(y,yc);
disp('The correlation coefficient is:')
rr=min(r);
disp(rr(1));
```

Example 7: In an experiment to verify Ohm's Law, the voltage and current acting on a resistor was measured and the following results obtained.

V	0	2	4	6	8	10
I	0	0.19	0.42	0.62	0.75	0.92

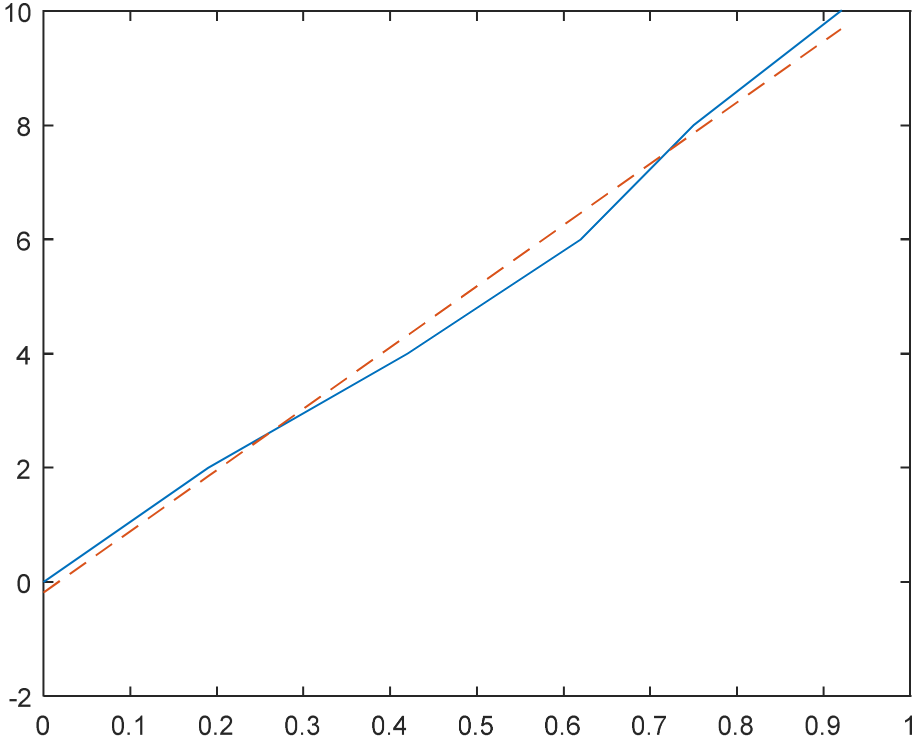
Assuming the law should be linear, find the best fit straight line.



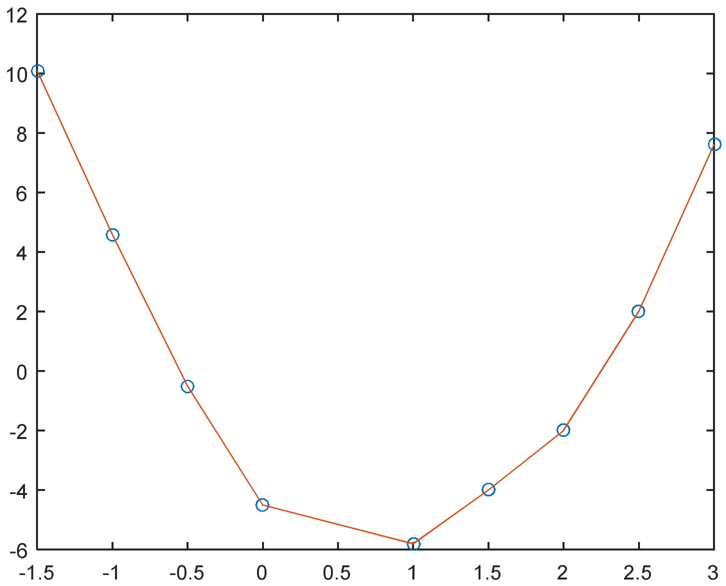
x=I= [0 .19 .42 .62 .75 .92]
y=V= [0 2 4 6 8 10]

In MATLAB
>> simplepolyfit
Type the x values as a vector enclosed within [ ]:
[0 .19 .42 .62 .75 .92]
Type the observed y values as a vector enclosed within [ ]:
[0 2 4 6 8 10]

coeffs =
  10.7261   -0.1843
The correlation coefficient is:
   0.9965



Example 8:
x= [-1.5 -1 -.5 0 1.0 1.5 2 2.5 3]
y= [10.1 4.6 -.5 -4.5 -5.8 -4 -2 2 7.6]



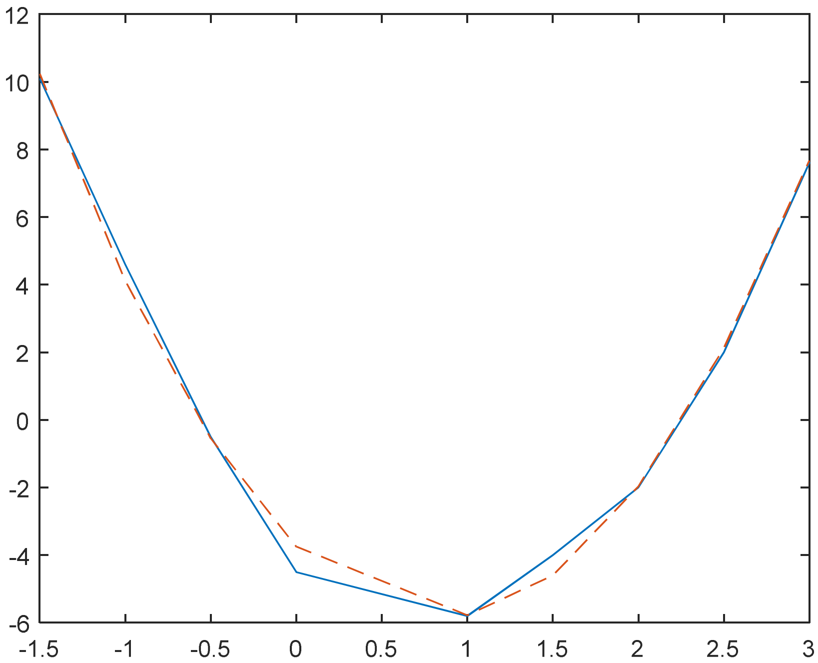
Fitted curve, Polynomial of order 2
y= f(x,y)=$2.94x^2-4.98x-3.81$

In MATLAB:
>> polyfitn

Type the x values as a vector enclosed within [ ]:
[-1.5 -1 -.5 0 1.0 1.5 2 2.5 3]
Type the observed y values as a vector enclosed within [ ]:
[10.1 4.6 -.5 -4.5 -5.8 -4 -2 2 7.6]
Type the desired polynomial degree:
2

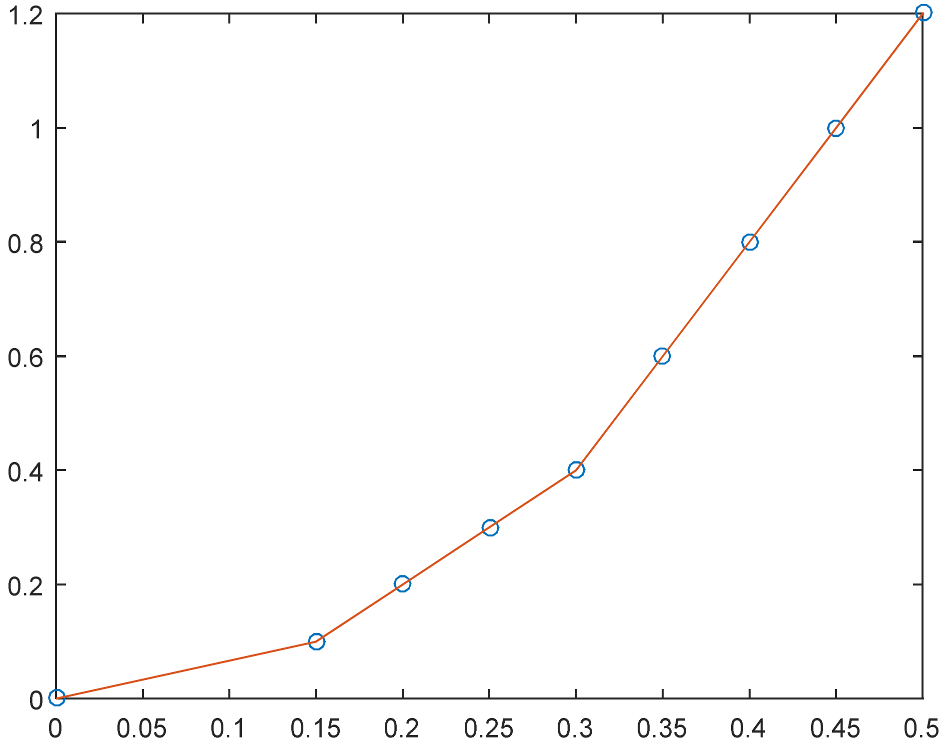
coeffs =
    2.9177   -4.9464   -3.7484

The correlation coefficient is:
    0.9976



Example 9: In an experiment to measure the distance a body moves in time t seconds when dropped from a height the following results were obtained. It is thought that the results should obey the law of gravity such that $s = gt^2/2$ where g is the gravitational constant. Obtain the best fit and determine the value of g.

time, t (seconds), x= [0 0.15 0.2 0.25 0.3 0.35 0.4 0.45 0.5]
distance, s (meters), y= [0 0.1 0.2 0.3 0.4 0.6 0.8 1.0 1.2]



For $s = gt^2/2$, we may provide an initial guess of a=1 and n=2.

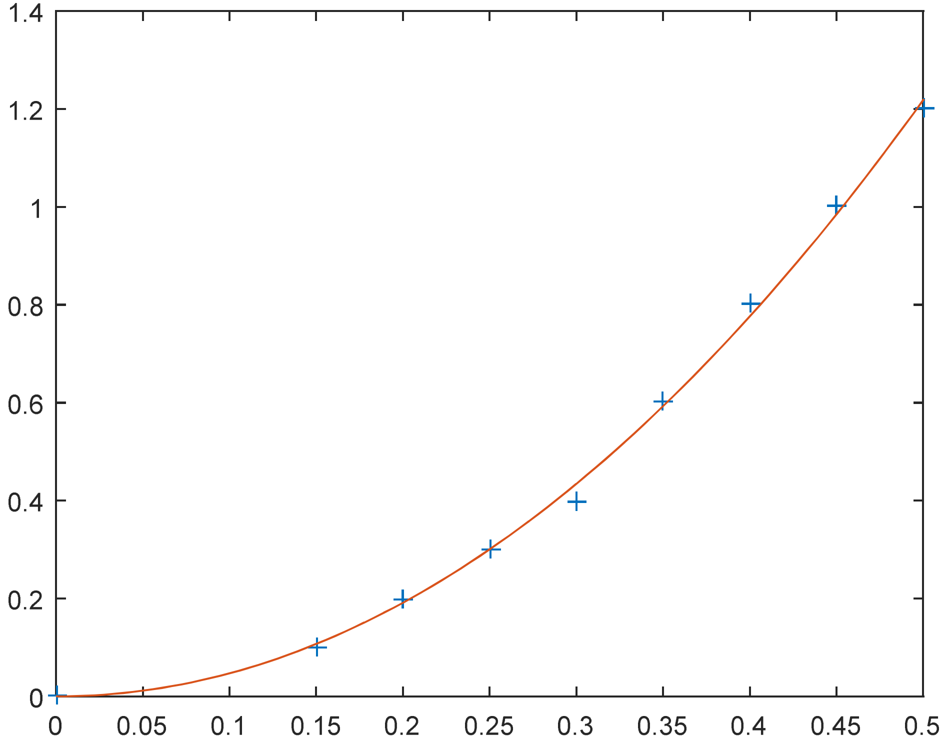
In MATLAB:
>> nonlinfit
Type the x values as a vector enclosed within [ ]:

[0 0.15 0.2 0.25 0.3 0.35 0.4 0.45 0.5]
Type the observed y values as a vector enclosed within [ ]:
[0 0.1 0.2 0.3 0.4 0.6 0.8 1.0 1.2]
Type an initial estimate for parameter a:
1
Type an initial estimate for parameter n:
2
The r^2 value for this fit is 0.998219

The script for nonlinfit is:

```
function nonlinfit
clear all
x=input('Type the x values as a vector enclosed within [ ]:\n');
y=input('Type the observed y values as a vector enclosed within [ ]:\n');
a=input('Type an initial estimate for parameter a:\n');
n=input('Type an initial estimate for parameter n:\n');
numpts=max(size(x));
zin(1)=1; %guess for first parameter
zin(2)=3; %guess for second parameter

zout=fminsearch(@(z) sumoferrs(z,x,y), zin);
xplot=x(1):(x(end)-x(1))/(10*numpts):x(end);
yplot=curve(xplot,zout);
plot(x,y,'+',xplot,yplot)
% The following lines attempt to assess the quality of the fit
datamean=mean(y);
errorsum=0;
othersum=0;
```

```
for  i=1:numpts
    errorsum=errorsum+(curve(x(i),zout)-y(i))^2;
    othersum=othersum+(datamean-y(i))^2;
end
rsquared=1-errorsum/othersum;
fprintf('The r^2 value for this fit is %f\n',rsquared)

function f=curve(x,z)

a=z(1);

n=z(2);

f=a*x.^n;

function f=sumoferrs(z, x, y)

f=sum((curve(x,z)-y).^2);
```



http://www.originlab.com/index.aspx?go=Products/Origin/DataAnalysis/CurveFitting
http://www.bioe.umd.edu/~artjohns/software/curvefit/curvefitting.pdf
https://en.wikipedia.org/wiki/Curve_fitting
http://blanchard.ep.wisc.edu/PublicMatlab/Fits/Fits.pdf

http://blanchard.ep.wisc.edu/PublicMatlab/
http://blanchard.ep.wisc.edu/PublicMatlab
http://www.mathworks.com/products/index.html;jsessionid=f43d14ef3fe57c4e80fc54cb2426
http://physics.info/curve-fitting/
https://ece.uwaterloo.ca/~dwharder/NumericalAnalysis/06LeastSquares/linear/
http://hyperphysics.phy-astr.gsu.edu/hbase/frict.html
http://hyperphysics.phy-astr.gsu.edu/hbase/mechanics/frictire.html#c1