Determinant Story

Determinant is a value associated with a square matrix. This is something to remember.

We can do without determinants but they do serve a useful purpose. It is a helpful concept which assists us in solving problems that recur frequently. Giving it a name simplifies matters. What mathematicians do is to solve problems efficiently and with style. Loving style is good. This story reveals that some values and calculations prove to be consistent and very useful, as well as very meaningful context-wise. We give them names and make them our friends. Or tools.

In the early days, before mathematicians discovered the elegance of matrices for resolving systems of linear equations, determinants were linked to systems of linear equations themselves. Their name is actually a give-away - it literally **determines** whether the system has a unique solution (nor over-constrained nor under-constrained). And the math showed that this value has to be everything but 0 to hold true. To have a unique solution. After these first baby steps, many mathematicians added to the foundation of the original *theory* and the concept of the determinant flourished as people solved more and more problems.

A determinant is used in many context-specific ways. In the context of its earliest usage, they were an indicator whether a system of linear equations has a **unique** solution. The condition? The value given the name of "determinant" has to be nonzero. As matrices became more useful over time and got many new upgrades, speaking figuratively - for example - we can represent linear transformations (a set of linear equations which respond to a few rules and transforms vectors from one coordinate system to another). Now, the pretty part of matrices in the geometric context is that if you multiply a point which you've transformed with a matrix A, you can undo that with an operation with its inverse A^{-1}. But there are cases when a matrix simply doesn't have a friend matrix which when multiplied together yields the n x n identity matrix - the equivalent of number 1.

Mathematicians found out that the good old concept of a determinant can be used as an indicator whether a matrix has an inverse. If it's 0, no inverse - sorry.

This is a simplification to get your feet wet. Just understand what the primary idea behind every concept in mathematics is. We don't pull them out of a hat, they knock on our doors when we work on problems. A fool shows them the way out by ignoring them, a wise man accepts their offerings. To be a bit poetic.

Determinant Examples

The determinant of a matrix is a **special number** that can be calculated from a square matrix.

A matrix is an array of numbers:

A Matrix
(This one has 2 Rows and 2 Columns)

The determinant of that matrix is (calculations are explained later):

3*6 − 8*4 = 18 − 32 = **−14**

What is it for?

The determinant tells us things about the matrix that are useful in systems of linear equations, helps us find the inverse of a matrix, is useful in calculus and more.

Symbol

The **symbol** for determinant is two vertical lines either side.

Example:

|A| means the determinant of the matrix **A**

(Exactly the same symbol as absolute value.)

Calculating the Determinant

First of all the matrix must be **square** (i.e. have the same number of rows as columns). Then it is just basic arithmetic. Here is how:

For a 2×2 Matrix

For a 2×2 matrix (2 rows and 2 columns):

The determinant is:

|A| = a*d - b*c
"The determinant of A equals a times d minus b times c"

It is easy to remember when you think of a cross:

Blue means positive (+a*d),

Red means negative (-b*c)

Example:

$$B = \begin{bmatrix} 4 & 6 \\ 3 & 8 \end{bmatrix}$$

|B| = 4*8 - 6*3

= 32-18

= 14

For a 3×3 Matrix

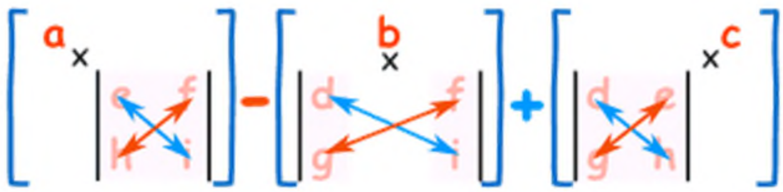
For a 3×3 matrix (3 rows and 3 columns):

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

The determinant is:

|A| = a*(e*i - f*h) - b*(d*i - f*g) + c(d*h - e*g)

It may look complicated, but **there is a pattern**:



To work out the determinant of a **3×3** matrix:

Multiply **a** by the **determinant of the 2×2 matrix** that is **not** in **a**'s row or column.

Likewise, for **b**, and for **c**

Add them up, but remember that **b** has a negative sign!

As a formula *(remember the vertical bars | | mean "determinant of")*:

$$|A| = a \cdot \begin{vmatrix} e & f \\ h & i \end{vmatrix} - b \cdot \begin{vmatrix} d & f \\ g & i \end{vmatrix} + c \cdot \begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

"The determinant of A equals a times the determinant of ... etc"

Example:

$$C = \begin{bmatrix} 6 & 1 & 1 \\ 4 & -2 & 5 \\ 2 & 8 & 7 \end{bmatrix}$$

|C| = 6*(-2*7 - 5*8) - 1*(4*7 - 5*2) + 1*(4*8 - -2*2)

= 6*(-54) - 1*(18) + 1*(36)

= **-306**

For 4×4 Matrices and Higher

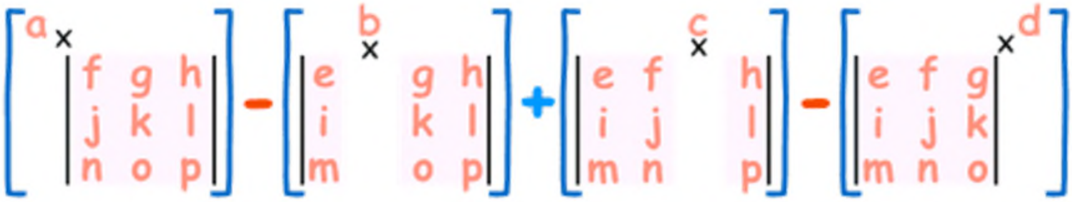
The pattern continues for 4×4 matrices:

plus a times the determinant of the matrix that is **not** in **a**'s row or column,

minus b times the determinant of the matrix that is **not** in **b**'s row or column,

plus c times the determinant of the matrix that is **not** in **c**'s row or column,

minus d times the determinant of the matrix that is **not** in **d**'s row or column,



As a formula:

$$|A| = a \cdot \begin{vmatrix} f & g & h \\ j & k & l \\ n & o & p \end{vmatrix} - b \cdot \begin{vmatrix} e & g & h \\ i & k & l \\ m & o & p \end{vmatrix} + c \cdot \begin{vmatrix} e & f & h \\ i & j & l \\ m & n & p \end{vmatrix} - d \cdot \begin{vmatrix} e & f & g \\ i & j & k \\ m & n & o \end{vmatrix}$$

Notice the + - + - pattern (+a... -b... +c... -d...). This is important to remember.

The pattern continues for 5×5 matrices and higher. Usually best to use a Matrix Calculator for those!

Not The Only Way

This method of calculation is called the "Laplace expansion" The pattern in it is easy to remember. But there are other methods (just so you know).

Summary

For a 2×2 matrix the determinant is **a*d - b*c**

For a 3×3 matrix multiply **a** by the **determinant of the 2×2 matrix** that is**not** in **a**'s row or column, likewise for **b** and **c**, but remember that **b** has a negative sign!

The pattern continues for larger matrices: multiply **a** by the **determinant of the matrix** that is **not** in **a**'s row or column, continue like this across the whole row, but remember the + - + - pattern.